ANALYTICITY OF LAYER POTENTIALS AND L^2 SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR DIVERGENCE FORM ELLIPTIC EQUATIONS WITH COMPLEX L^∞ COEFFICIENTS

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A . We consider divergence form elliptic operators of the form $L = -\operatorname{div} A(x)\nabla$, defined in $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$, $n \geq 2$, where the L^∞ coefficient matrix A is $(n+1) \times (n+1)$, uniformly elliptic, complex and t-independent. We show that for such operators, boundedness and invertibility of the corresponding layer potential operators on $L^2(\mathbb{R}^n) = L^2(\partial\mathbb{R}^{n+1}_+)$, is stable under complex, L^∞ perturbations of the coefficient matrix. Using a variant of the Tb Theorem, we also prove that the layer potentials are bounded and invertible on $L^2(\mathbb{R}^n)$ whenever $A(x)$ is real and symmetric (and thus, by our stability result, also when A is complex, $\|A - A^0\|_\infty$ is small enough and A^0 is real, symmetric, L^∞ and elliptic). In particular, we establish solvability of the Dirichlet and Neumann (and Regularity) problems, with L^2 (resp. $\dot{L}^2_1$) data, for small complex perturbations of a real symmetric matrix. Previously, L^2 solvability results for complex (or even real but non-symmetric) coefficients were known to hold only for perturbations of constant matrices (and then only for the Dirichlet problem), or in the special case that the coefficients $A_{j,n+1} = 0 = A_{n+1,j}$, $1 \leq j \leq n$, which corresponds to the Kato square root problem.

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1. I , ,

In this paper, we consider the solvability of boundary value problems for divergence form complex coefficient equations $Lu = 0$, where

$$L = -\operatorname{div} A\nabla \equiv -\sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i}\left(A_{i,j}\,\frac{\partial}{\partial x_j}\right)$$

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is defined in $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$, $n \geq 2$ (we use the notational convention that $x_{n+1} = t$), and where $A = A(x)$ is an $(n+1)\times(n+1)$ matrix of complex-valued L^∞ coefficients, defined on $\mathbb{R}^n$ (i.e., independent of the t variable) and satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \Re e\,\langle A(x)\xi,\xi\rangle \equiv \Re e \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\bar{\xi_i}, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda, \tag{1.1}$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$. The divergence form equation is interpreted in the weak sense, i.e., we say that $Lu = 0$ in a domain Ω if $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int A\nabla u \cdot \overline{\nabla\Psi} = 0$$

for all complex valued $\Psi \in C_0^\infty(\Omega)$.

The boundary value problems that we consider are classical. To state them, we first recall the definitions of the non-tangential maximal operators $N_*, \widetilde{N}_*$. Given $x_0 \in \mathbb{R}^n$, define the cone $\gamma(x_0) = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x_0 - x| < t\}$. Then for U defined in $\mathbb{R}^{n+1}_+$,

$$N_*U(x_0) \equiv \sup_{(x,t)\in\gamma(x_0)} |U(x,t)|, \quad \widetilde{N}_*U(x_0) \equiv \sup_{(x,t)\in\gamma(x_0)} \left(\fint\!\!\fint_{\substack{|x-y|<t \\ |t-s|<t/2}} |U(y,s)|^2 dyds \right)^{\frac{1}{2}}.$$

Here, and in the sequel, the symbol $\fint$ denotes the mean value, i.e., $\fint_E f \equiv |E|^{-1}\int_E f$. We use the notation $u \to f$ *n.t.* to mean that for *a.e.* $x \in \mathbb{R}^n$, $\lim_{(y,t)\to(x,0)} u(y,t) = f(x)$, where the limit runs over $(y,t) \in \gamma(x)$.

We shall consider the Dirichlet problem[1]

$$\text{(D2)} \qquad \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times (0,\infty)\} \\ \lim_{t\to 0} u(\cdot,t) = f \text{ in } L^2(\mathbb{R}^n) \text{ and } n.t. \\ \sup_{t>0} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty, \end{cases}$$

the Neumann problem[2]

$$\text{(N2)} \qquad \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \frac{\partial u}{\partial \nu}(x,0) \equiv -\sum_{j=1}^{n+1} A_{n+1,j}(x)\frac{\partial u}{\partial x_j}(x,0) = g(x) \in L^2(\mathbb{R}^n) \\ \widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n), \end{cases}$$

and the Regularity problem

$$\text{(R2)} \qquad \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ u(\cdot,t) \to f \in \dot{L}^2_1(\mathbb{R}^n)\, n.t. \\ \widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n). \end{cases}$$

Our solutions will be unique among the class of solutions satisfying the stated L^2 bounds (in the case of (N2) and (R2), this uniqueness will hold modulo constants). The homogeneous Sobolev space $\dot{L}^2_1$ is defined as the completion of C_0^∞ with respect to the seminorm $\|\nabla F\|_2$. For $n \geq 3$ this space can be identified (modulo constants) with the space $I_1(L^2) \equiv \Delta^{-1/2}(L^2) \subset L^{2^*}$, where $2^* \equiv 2n/(n-2)$; for $n = 2$, the identification with $I_1(L^2)$ is

[1]Our uniform L^2 estimate for solutions of (D2) can be improved to an L^2 bound for N_*u, given certain L^p estimates for the layer potentials. The fourth named author and M. Mitrea will present the L^p theory in a forthcoming publication. In the present paper, we shall be content with a weak-L^2 bound for N_*u.

[2]We shall elaborate in section 4 the precise nature by which the co-normal derivative assumes the prescribed data.

still valid, but in that case the fractional integral $I_1 f$ must itself be defined modulo constants for $f \in L^2$, and the space embeds in BMO.

We remark that for the class of operators that we consider, solvability of these boundary value problems in the half-space may readily be generalized to the case of domains given by the region above a Lipschitz graph, and even to the case of star-like Lipschitz domains. We shall return to this point later. We shall also discuss later the significance of our assumption that the coefficients are t-independent.

In order to state our main results, we shall need to recall a few definitions and facts. We say that u is locally Hölder continuous in a domain Ω if there is a constant C and an exponent $\alpha > 0$ such that for any ball $B = B(X, R)$, of radius R, whose concentric double $2B \equiv B(X, 2R)$ is contained in Ω, we have that

$$|u(Y) - u(Z)| \leq C \left(\frac{|Y - Z|}{R} \right)^{\alpha} \left(\fint_{2B} |u|^2 \right)^{\frac{1}{2}}, \tag{1.2}$$

whenever $Y, Z \in B$. Observe that any u satisfying (1.2) also satisfies Moser's "local boundedness" estimate [M]

$$\sup_{Y \in B} |u(Y)| \leq C \left(\fint_{2B} |u|^2 \right)^{\frac{1}{2}}. \tag{1.3}$$

By the classical De Giorgi-Nash Theorem [DeG, N], (1.2) and hence also (1.3) hold, with C and α depending only on dimension and the ellipticity parameters, whenever u is a solution of $Lu = 0$ in $\Omega \subseteq \mathbb{R}^{n+1}$, if *in addition* the coefficient matrix A is real (for this result, it need not be t-independent). Moreover, it is shown in [A] (see also [AT, HK]), that property (1.2) is stable under complex, L^∞ perturbations.

We now recall the method of layer potentials. For L as above, let Γ, Γ^* denote the fundamental solutions[3] for L and L^* respectively, in $\mathbb{R}^{n+1}$, so that

$$L_{x,t}\,\Gamma(x, t, y, s) = \delta_{(y,s)}, \quad L^*_{y,s}\,\Gamma^*(y, s, x, t) \equiv L^*_{y,s}\,\overline{\Gamma(x, t, y, s)} = \delta_{(x,t)},$$

where δ_X denotes the Dirac mass at the point X, and L^* is the hermitian adjoint of L. By the t-independence of our coefficients, we have that

$$\Gamma(x, t, y, s) = \Gamma(x, t - s, y, 0). \tag{1.4}$$

We define the single and double layer potential operators, by

$$\begin{aligned} \mathcal{S}_t f(x) &\equiv \int_{\mathbb{R}^n} \Gamma(x, t, y, 0)\, f(y)\, dy, \quad t \in \mathbb{R} \\ \mathcal{D}_t f(x) &\equiv \int_{\mathbb{R}^n} \overline{\partial_{\nu^*} \Gamma^*(y, 0, x, t)}\, f(y)\, dy, \quad t \neq 0, \end{aligned} \tag{1.5}$$

where ∂_{ν^*} is the adjoint exterior conormal derivative; i.e., if A^* denotes the hermitian adjoint of A, then

$$\partial_{\nu^*}\Gamma^*(y, 0, x, t) = -\sum_{j=1}^{n+1} A^*_{n+1,j}(y) \frac{\partial \Gamma^*}{\partial y_j}(y, 0, x, t) = -e_{n+1} \cdot A^*(y) \nabla_{y,s} \Gamma^*(y, s, x, t)\,|_{s=0}$$

[3]See [HK2] for a construction of the fundamental solution.

(recall that $y_{n+1} = s$). We define (loosely[4], for the moment) boundary singular integrals

$$
\begin{aligned}
Kf(x) &\equiv \text{“}p.v.\text{”} \int_{\mathbb{R}^n} \overline{\partial_{\nu^*}\Gamma^*(y,0,x,0)}\, f(y)\, dy \\
\widetilde{K} f(x) &\equiv \text{“}p.v.\text{”} \int_{\mathbb{R}^n} \frac{\partial \Gamma}{\partial \nu}(x,0,y,0)\, f(y)\, dy
\end{aligned} \tag{1.6}
$$

where $\frac{\partial}{\partial\nu}$ denotes the exterior conormal derivative in the (x,t) variables. Classically, $\widetilde{K}$ is often denoted K^*, but we avoid this notation here as $\widetilde{K}$ need not be the adjoint of K unless L is self-adjoint. Rather, for us, K^*, S^* and $\mathcal{D}^*$ will denote the analogues of K, S and $\mathcal{D}$ corresponding to L^* (although sometimes we shall write K^{L^*}, etc., when we wish to emphasize the dependence on a particular operator), and we use the notation $adj\,(T)$ to denote the Hermitian adjoint of an operator T acting in $\mathbb{R}^n$. With these conventions, we have that $\widetilde{K} = adj\,(K^*)$, as the reader may verify. We apologize for this departure from tradition, but the context of complex coefficients seems to require it.

For sufficiently smooth coefficients, the following “jump relation” formulae have been established in [MMT]. We defer to Section 4 our discussion of the jump formulae, and the nature of their “non-tangential” realization, in the non-smooth case. We have

$$
\mathcal{D}_{\pm s} f \;\to\; \left(\mp\frac{1}{2}I + K\right) f \tag{1.7}
$$

$$
(\nabla S_t)\,|_{t=\pm s} f \;\to\; \mp\frac{1}{2}\cdot\frac{f(x)}{A_{n+1,n+1}(x)} e_{n+1} + \mathcal{T} f, \tag{1.8}
$$

(these convergence statements must be interpreted properly - see Section 4) where

$$
\mathcal{T} f(x) \equiv \text{“}p.v.\text{”} \int_{\mathbb{R}^n} \nabla\Gamma(x,0,y,0) f(y) dy. \tag{1.9}
$$

Then, as usual[5], one obtains solvability of (D2) in the upper (resp. lower) half space by establishing boundedness on $L^2(\mathbb{R}^n)$ of $f \to \mathcal{D}_{\pm t} f$, uniformly in t, and invertibility of $-\frac{1}{2}I+K$ (resp. $\frac{1}{2}I+K$). Similarly, solvability of (N2) and (R2) follows from L^2 boundedness of $f \to \widetilde{N}_*(\nabla S_{\pm t} f)$, and (for (N2)) invertibility on L^2 of $\pm\frac{1}{2}I+\widetilde{K}$, and (for (R2)) invertibility of the mapping $S_0 = S_t|_{t=0} : L^2(\mathbb{R}^n) \to \dot{L}^2_1(\mathbb{R}^n)$. We now set some convenient terminology: we shall say that an operator L for which all of the above hold has “Bounded and Invertible Layer Potentials”. If *in addition* we have the square function estimate

$$
\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \left|t\partial_t^2 S_t f(x)\right|^2 \frac{dxdt}{|t|} \le C\|f\|_2^2, \tag{1.10}
$$

then we shall say that L has “Good Layer Potentials”. Finally, we shall refer to the constant in (1.10), together with all of the constants arising in the estimates for the boundedness and invertibility of the layer potentials, collectively as the “Layer Potentials Constants” for L.

In this paper, we prove the following theorems. In the sequel we assume always that our $(n+1)\times(n+1)$ coefficient matrices are $t-$independent, complex, and satisfy the ellipticity condition (1.1) and the De Giorgi-Nash-Moser estimates (1.2) and (1.3).

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[4]For non-smooth coefficients, some care should be taken to define the “principal value” operators on the boundary - see Section 4.

[5]In the setting of non-smooth coefficients, some rather extensive preliminaries are required in order to apply the layer potential method to obtain solvability; see Section 4.

Theorem 1.11. *Suppose that $L_0 = -\operatorname{div} A^0\nabla$ and $L_1 = -\operatorname{div} A^1\nabla$ are operators of the type described above, and that solutions u_0, w_0 of $L_0u_0 = 0$, $L_0^*w_0 = 0$ satisfy the De Giorgi-Nash-Moser estimates* (1.2) *and* (1.3). *Suppose also that L_0 and L_0^* have "Good Layer Potentials". Then L_1 and L_1^* have Good Layer Potentials, provided that*

$$\|A^0 - A^1\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon_0,$$

where ϵ_0 is sufficiently small depending only on dimension and on the various constants associated to L_0 and L_0^*, specifically: the ellipticity parameters, the De Giorgi-Nash-Moser constants (1.2) *and* (1.3)*, and the Layer Potential Constants.*

We observe that it is not clear whether the property that L has "Good Layer Potentials" is preserved under regularization of the coefficients. For this reason, we shall be forced to prove Theorem 1.11 without recourse to the usual device of making an *a priori* assumption of smooth coefficients. We also note that we shall use the invertibility of the layer potentials associated to L_0 and L_0^* even to establish the boundedness of the layer potentials associated to L_1 (see Section 7 below).

Theorem 1.12. *Suppose that $L = -\operatorname{div} A\nabla$ is an operator of the type defined above, and in addition, suppose that A is real and symmetric. Then L has Good Layer Potentials, and its Layer Potential Constants depend only on dimension and on the ellipticity parameters in* (1.1).

We remark that while Theorem 1.12 yields in particular the solvability of (D2), (N2) and (R2) in the case that A is real and symmetric, it is only the fact that this solvability is obtainable via layer potentials that is new here, the solvability of (D2) having been previously obtained by Jerison and Kenig [JK1], and that of (N2) and (R2) by Kenig and Pipher [KP], without the use of layer potentials. The essential missing ingredient had been the boundedness of the layer potentials.

The previous two theorems are our main results. As corollaries, we obtain

Theorem 1.13. *Suppose that $L_1 = -\operatorname{div} A^1\nabla$ is an operator of the type defined above, and that $\|A^1 - A^0\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon_0$, for some real, symmetric, t-independent uniformly elliptic matrix $A^0 \in L^\infty(\mathbb{R}^n)$. Then* (D2), (N2) *and* (R2) *are all solvable for L_1, provided that ϵ_0 is sufficiently small, depending only on dimension and the ellipticity parameters for A^0. The solution of* (D2) *is unique among the class of solutions u for which $\sup_{t>0}\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$, and the solutions of* (N2) *and* (R2) *are unique modulo constants among the class of solutions for which $\widetilde{N}_*(\nabla u) \in L^2$.*

Theorem 1.14. *The conclusion of Theorem 1.13 holds also in the case that $\|A^1 - A^0\|_\infty$ is sufficiently small, where A^0 is now a constant, elliptic complex matrix.*

The last theorem follows from Theorem 1.11, and the fact that constant coefficient operators have Good Layer Potentials (see the appendix, Section 10).

We note that by a standard device, Theorems 1.11, 1.12 and 1.13 all extend readily to the case where $\Omega = \{(x,t) : t > F(x)\}$, with F Lipschitz. Indeed, by "pulling back" under the mapping $\rho : \mathbb{R}^{n+1}_+ \to \Omega$ defined by

$$\rho(x,t) = (x, F(x) + t),$$

we may reduce to the case of the half-space. The pull-back operators are of the same type, and, in particular, the coefficients remain t-independent. Moreover, if the original coefficients are real and symmetric, then so are those of the pull-back operator. In this setting, the parameter ϵ_0 will also depend on $\|\nabla F\|_\infty$. In addition, our results may be

further extended to the setting of star-like Lipschitz domains (which would seem to be the most general setting in which the notion of "radial independence" of the coefficients makes sense). The idea is to use a partition of unity argument, as in [MMT], to reduce to the case of a Lipschitz graph. We omit the details.

Let us now briefly review the history of work in this area, which falls broadly into two categories, depending on whether or not the t-independent coefficient matrix is self-adjoint. We discuss the former category first, and we mention only the case of a single equation, although results for certain constant coefficient self-adjoint systems in a Lipschitz domain are known, see e.g. [K, K2] for further references. (Moreover, the present setting of complex coefficients may be viewed in the context of 2×2 systems, and indeed this provides part of our motivation to consider the complex case). For Laplace's equation in a Lipschitz domain, the solvability of (D2) was obtained by Dahlberg [D], and that of (N2) and (R2) by Jerison and Kenig [JK2]; solvability of the same problems via harmonic layer potentials is due to Verchota [V], using the deep result of Coifman, McIntosh and Meyer [CMcM] concerning the L^2 boundedness of the Cauchy integral operator on a Lipschitz curve. The results of [V] and [CMcM] are subsumed in our Theorem 1.12 via the pull-back mechanism discussed above. Moreover, as mentioned above, for A real, symmetric and t-independent, the solvability of (D2) was obtained in [JK1], and that of (N2) and (R2) in [KP], but those authors did not use layer potentials. The case of real symmetric coefficients with some smoothness has been treated via layer potentials in [MMT].

In the "non self-adjoint" setting, previous results had been obtained in three special cases. First, it was known that (D2) is solvable for small, complex perturbations of *constant* elliptic matrices. This is due to Fabes, Jerison and Kenig [FJK] via the method of multilinear expansions. To our knowledge, (R2) and (N2) had not been treated in this setting.

Second, one has solvability of (D2), (N2) and (R2) in the special case that the matrix A is of the "block " form

$$\left[\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \\ \hline 0\cdots 0 & 1 \end{array}\right] \tag{1.15}$$

where $B = B(x)$ is a $n\times n$ matrix. In this case, (D2) is an easy consequence of the semigroup theory, while (R2) amounts to solving the Kato square root problem for the n-dimensional operator

$$J = -\operatorname{div}_x B(x)\nabla_x,$$

and (N2) amounts to L^2 boundedness of the Riesz transforms $\nabla J^{-\frac{1}{2}}$ (equivalently, to solving the Kato problem for the adjoint operator $adj\,(J)$). Moreover, the boundedness of the Riesz transform $\nabla J^{-\frac{1}{2}}$ can also be interpreted as the statement that the single layer potential is bounded from L^2 into $\dot{L}^2_1$. These results were obtained in [CMcM] ($n = 1$), [HMc] ($n = 2$), [AHLT] (when B is a perturbation of a real, symmetric matrix), [HLMc] (when the kernel of the heat semi-group e^{-tJ} has a Gaussian upper bound) and [AHLMcT] in general[6].

[6] We remark that Theorem 1.11 may be combined with these results for block matrices (1.15) to allow perturbations of the block case, but we do not pursue this point here; see, however, [AAH], where this is done without imposing De Giorgi-Nash-Moser bounds, and where also extensions of Theorems 1.13 and 1.14 will be presented, via the development of a functional calculus for certain Dirac type operators.

Third, Kenig, Koch, Pipher and Toro [KKPT] have obtained solvability of (Dp) (the problems (Dp), (Np) and (Rp) are defined analogously to (D2), (N2) and (R2), but with L^2 bounds replaced by L^p) in the case $n = 1$ (that is, in $\mathbb{R}^2_+$), for p sufficiently large depending on L, in the case that $A(x)$ is real, but non-symmetric. Moreover, they construct a family of examples in $\mathbb{R}^2_+$ in which solvability of (Dp) may be destroyed for any specified p by taking $A(x)$ to be an appropriate perturbation of the 2×2 identity matrix. Very recently, in the same setting of real, non-symmetric coefficients in two dimensions (that is, in $\mathbb{R}^2_+$), Kenig and Rule [KR] have obtained solvability of (Nq) and (Rq), where q is dual to the [KKPT] exponent. Their result uses boundedness, but not invertibility, of the layer potentials.

The main purpose, then, of the present paper is to develop, to the extent possible, an L^2 theory of boundary value problems for *full* coefficient matrices with complex (including also real, not necessarily symmetric) entries. In fact, in the setting of L^2 solvability with t-independent coefficients, the counter-example of [KKPT] shows that our perturbation results are in the nature of best possible.

A word about t-independence is in order. It has been observed by Caffarelli, Fabes and Kenig [CFK] that some regularity in the transverse direction is necessary, in order to deduce solvability of (D2). More precisely, they show that given any function $\omega(\tau)$ with $\int_0^1 (\omega(\tau))^2 d\tau/\tau = +\infty$, there exists a real, symmetric elliptic matrix $A(x, t)$, whose modulus of continuity in the t direction is controlled by ω, but for which the corresponding elliptic-harmonic measure and the Lebesque measure on the boundary are mutually singular. On the other hand, it is shown in [FJK] that (D2) does hold, assuming that the transverse modulus of continuity $\omega(\tau) \equiv \sup_{x\in\mathbb{R}^n,\, 0<t<\tau} |A(x, t) - A(x, 0)|$ satisfies the square Dini condition $\int_0^1 (\omega(\tau))^2 d\tau/\tau < \infty$, provided that $A(x, 0)$ is sufficiently close to a constant matrix A_{const}. It seems likely that the methods of the present paper would allow us to obtain a similar result, but with the constant matrix A_{const} replaced by an L^∞ matrix $A^0(x)$ satisfying the hypotheses of Theorem 1.11 (in particular, real, symmetric). However, we have not pursued this variant here, in part because we conjecture that somewhat sharper estimates should be true. To explain this point of view, we recall that a more refined, scale invariant version of the square Dini condition has been introduced by R. Fefferman, Kenig and Pipher [FKP], and Kenig and Pipher [KP, KP2], to prove perturbation results in which one assumes (roughly) that $|A^1(x, t) - A^0(x, t)|^2 \frac{dxdt}{t}$ is a Carleson measure (actually, their condition is slightly stronger, but in the same spirit). Note that this condition requires that $A^1 = A^0$ on the boundary. Our work provides a complement to [FKP] and [KP, KP2], in that we allow the coefficients to differ at the boundary. At present, the results of [FKP] and [KP, KP2] apply only to the case of real coefficients. It is an interesting open problem to extend the theorems of [FKP] and [KP, KP2] to the case of complex coefficients, even in the case of small Carleson norm. Given such an extension, along with our results here, one could specialize to the case $A^1(x, t) = A(x, t)$, $A^0(x, t) = A(x, 0)$, with $A(x, 0)$ close enough to a "good" (e.g., real, symmetric) matrix, to obtain a rather complete picture of the situation for L^2 solvability.

Let us now set some notation that will be used throughout the paper. We shall use div and ∇ to denote the full $n + 1$ dimensional divergence and gradient, respectively. At times, we shall need to consider the n-dimensional gradient and divergence, acting only in x, and these we denote either by $\nabla_{\|}$ and $\mathrm{div}_{\|}$, or by ∇_x and div_x; i.e.

$$\nabla_{\|} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right) = \nabla_x$$

and for $\mathbb{R}^n$-valued $\vec{w}$, $\operatorname{div}_{\|} \vec{w} \equiv \nabla_{\|} \cdot \vec{w}$. Similarly, given an $(n+1) \times (n+1)$ matrix A, we shall let $A_{\|}$ denote the $n \times n$ sub-matrix with entries $(A_{\|})_{i,j} \equiv A_{i,j}, \quad 1 \leq i, j \leq n$, and we define the corresponding elliptic operator acting in $\mathbb{R}^n$ by

$$L_{\|} \equiv -\operatorname{div}_x A_{\|} \nabla_x .$$

We shall also use the notation

$$D_j \equiv \frac{\partial}{\partial x_j} = \partial_{x_j}, \quad 1 \leq j \leq n+1$$

bearing in mind that $x_{n+1} = t$. Points in $\mathbb{R}^{n+1}$ may sometimes be denoted by capital letters, e.g. $X = (x,t)$, $Y = (y,s)$. Balls in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ will be denoted respectively by $B(X,r) \equiv \{Y : |X - Y| < r\}$ and $\Delta_r(x) \equiv \{y : |x - y| < r\}$. We shall often encounter operators whose kernels involve derivatives applied to the second set of variables in the fundamental solution $\Gamma(x,t,y,s)$. We shall indicate this by grouping the operators with appropriate parentheses, thus:

$$(S_t \nabla)\, f(x) \equiv \int_{\mathbb{R}^n} \nabla_{y,s} \Gamma(x,t,y,s)\,|_{s=0}\, f(y)\, dy.$$

Hence, one then has

$$(S_t \nabla_{\|}) \cdot \vec{f} = -S_t \left(\operatorname{div}_{\|} \vec{f} \right), \quad (S_t D_{n+1}) = -\partial_t S_t,$$

where in the second identity we have used (1.4)

Given a cube Q, we denote the side length of Q by $\ell(Q)$. Furthermore, given a positive number r, we let rQ denote the concentric cube with side length $r\ell(Q)$.

We shall use P_t to denote a nice approximate identity, acting on functions defined on $\mathbb{R}^n$; i.e. $P_t f(x) = \phi_t * f$, where $\phi_t(x) = t^{-n}\phi(x/t)$, $\phi \in C_0^\infty(\{|x| < 1\})$, $0 \leq \phi$ and $\int_{\mathbb{R}^n} \phi = 1$.

Following [FJK], we introduce a convenient norm for dealing with square functions (although we warn the reader that our measure differs from that used in [FJK]):

$$|||F|||_{\pm} \equiv \left(\iint_{\mathbb{R}^{n+1}_{\pm}} |F(x,t)|^2 \frac{dxdt}{|t|} \right)^{\frac{1}{2}}, \quad |||F|||_{all} \equiv \left(\iint_{\mathbb{R}^{n+1}} |F(x,t)|^2 \frac{dxdt}{|t|} \right)^{\frac{1}{2}}.$$

For a family of operators U_t, we write

$$|||U_t|||_{+,op} \equiv \sup_{\|f\|_{L^2(\mathbb{R}^n)}=1} |||U_t f|||_{+},$$

and similarly for $|||\cdot|||_{-,op}$ and $|||\cdot|||_{all,op}$. Sometimes, we may drop the "+" sign when it is clear that we are working in the upper $\frac{1}{2}$-space. As usual, we allow generic constants C to depend upon dimension and ellipticity, and, in the proof of the perturbation result, upon the constants associated to the "good" operator L_0. Specific constants, still depending on the same parameters, will be denoted C_1, C_2, etc..

The paper is organized as follows. In sections 2 and 3, we prove some useful technical estimates. In section 4 we discuss the boundary behavior and uniqueness of our solutions. The next five sections are the heart of the matter, in which we prove Theorem 1.11 (sections 5, 6 and 7), and Theorem 1.12 (sections 8 and 9). Section 10 is an appendix, in which we briefly discuss the constant coefficient case.

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2. S D G -N -M

Throughout this section, and throughout the rest of the paper, we suppose always that our differential operators satisfy our "standard assumptions": that is, divergence form elliptic, with ellipticity parameters λ and Λ, defined in $\mathbb{R}^{n+1}$, $n \geq 2$, with complex coefficients that are bounded, measurable and $t-$independent; moreover, we suppose that solutions of $Lu = 0$ satisfy the De Giorgi-Nash-Moser estimates (1.2) and (1.3). We now prove some technical estimates using rather familiar arguments. In the sequel, Γ will denote the fundamental solution of L, and we set

$$K_{m,t}(x,y) \equiv (\partial_t)^{m+1}\Gamma(x,t,y,0) \tag{2.1}$$

Lemma 2.2. *Suppose that L and L^* satisfy the "standard assumptions" as above. Then there exists a constant C_1 depending only on dimension, ellipticity and* (1.2) *and* (1.3), *such that for every integer $m \geq -1$, for all $t \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$, we have*

$$\left|K_{m,t}(x,y)\right| \leq C C_1^{m^2} (|t| + |x-y|)^{-n-m} \tag{2.3}$$

$$\left|\left(\mathbb{D}^h K_{m,t}(\cdot,y)\right)(x)\right| + \left|\left(\mathbb{D}^h K_{m,t}(x,\cdot)\right)(y)\right| \leq C C_1^{m^2} \frac{|h|^\alpha}{(|t| + |x-y|)^{n+m+\alpha}}, \tag{2.4}$$

whenever $2|h| \leq |x-y|$ or $|h| < 20|t|$, for some $\alpha > 0$, where $\left(\mathbb{D}^h f\right)(x) \equiv f(x+h) - f(x)$.

Sketch of proof. The case $m = -1$ of (2.3) follows from its parabolic analogue in [AT], Section 1.4; alternatively, the reader may consult [HK2] for a direct proof in the elliptic case. The case $m = 0$ may be treated by applying (1.3) to the solution $u(x,t) = \partial_t\Gamma(x,t,y,0)$ in the ball $B((x,t), R/2)$, with $R = \sqrt{|t|^2 + |x-y|^2}$, and then using Caccioppoli's inequality to reduce to the case $m = -1$. The case $m > 0$ is obtained by iterating the previous argument, and (2.4) follows from (1.2) and (2.3). □

We remark that, by taking more care with the Caccioppoli argument, using a ball of appropriately chosen radius $c_m R$ rather than $R/2$, one may obtain the natural growth bound $m! C_1^m$ in (2.3) and (2.4). We leave the details to the interested reader.

Lemma 2.5. *Suppose that L, L^* satisfy the standard assumptions. Then, there exists a constant C_2, and for each $\rho > 1$ a constant C_ρ, such that for every cube $Q \subseteq \mathbb{R}^n$, for all $x \in Q$, and for all integers $k \geq 1$ and $m \geq -1$, we have*

(i) $\int_{2^{k+1}Q\setminus 2^kQ} \left|(2^k\ell(Q))^m (\partial_t)^{m+1} \nabla_y\Gamma(x,t,y,0)\right|^2 dy \leq C C_2^{m^2} (2^k\ell(Q))^{-n-2}, \quad \forall t \in \mathbb{R}$

(ii) $\int_{2Q} \left|\ell(Q)^m (\partial_t)^{m+1} \nabla_y\Gamma(x,t,y,0)\right|^2 dy \leq C_\rho^{m^2+1} \ell(Q)^{-n-2}, \quad \frac{\ell(Q)}{\rho} < |t| < \rho\, \ell(Q).$

Proof. We first suppose that $A \in C^\infty$; we shall remove this restriction at the end of the proof. Of course, our quantitative bounds will not depend on smoothness. Let us consider estimate (i) first. We shall actually prove that for C_2 large enough we have

$$\sum_{m=0}^{\infty} C_2^{-m^2} \left\|(2^k\ell(Q))^m (\partial_t)^{m+1} \nabla_y\Gamma(x,t,\cdot,0)\right\|^2_{L^2(2^{k+1}Q\setminus 2^kQ)} \leq C(2^k\ell(Q))^{-n-2}. \tag{2.6}$$

Fix $x \in Q$. Let $\varphi_k \in C_0^\infty$, $\varphi_k \equiv 1$ on $2^{k+1}Q\setminus 2^kQ$, $\operatorname{supp}\varphi_k \subset \frac{3}{2}2^{k+1}Q\setminus\frac{3}{2}2^{k-1}Q$, with

$$\|\nabla\varphi_k\|_\infty \leq C(2^k\ell(Q))^{-1}.$$

We observe that

$$\begin{aligned} I_m &\equiv \int \left|(\partial_t)^{m+1}\,\nabla_y\Gamma(x,t,y,0)\right|^2 \varphi_k^2(y)dy \\ &\le C\,\Re e\int A_\|^*\,\nabla_y\,(\partial_t)^{m+1}\,\Gamma(x,t,y,0)\cdot\overline{\nabla_y\,(\partial_t)^{m+1}\,\Gamma(x,t,y,0)}\varphi_k^2(y)\,dy \end{aligned}$$

(where $A_\|^*$ is the adjoint of the $n\times n$ matrix $A_\|$ defined by $(A_\|)_{ij} = A_{ij}$, $1 \le i, j \le n$)

$$\begin{aligned} &= C\,\Re e\int (L_\|^*)_y\,(\partial_t)^{m+1}\,\Gamma(x,t,y,0)\,\overline{(\partial_t)^{m+1}\,\Gamma(x,t,y,0)}\,\varphi_k^2(y)\,dy \\ &\quad - C\,\Re e\int A_\|^*\,\nabla_y\,(\partial_t)^{m+1}\,\Gamma(x,t,y,0)\,\overline{(\partial_t)^{m+1}\,\Gamma(x,t,y,0)}\cdot\nabla_y\varphi_k^2(y)\,dy \\ &= I'_m + I''_m, \end{aligned}$$

where $L_\|^* \equiv -\operatorname{div}_x A_\|^*\nabla_x$. For each integer $m \ge -1$, define

$$a_m = a_m(x) \equiv \|(2^k\ell(Q))^m\,(D_{n+1})^{m+1}\,\nabla_y\Gamma(x,t,\cdot,0)\varphi_k\|_2 = \left(2^k\ell(Q)\right)^m I_m^{1/2}.$$

Since $\Gamma(x,t,\cdot,\cdot)$ is a solution of L^* away from x,t, we have that

$$(L_\|^*)_y\,\Gamma(x,t,y,0) = \sum_{i=1}^n D_iA^*_{i,n+1}D_{n+1}\Gamma + \sum_{j=1}^{n+1} A^*_{n+1,j}\cdot D_{n+1}D_j\Gamma,$$

where in the second term we have used t-independence. We designate the respective contribution of these two terms to I'_m by $I'_{m,1}$ and $I'_{m,2}$. Now,

$$\begin{aligned} |I'_{m,2}| &\le C\int |\nabla_{y,s}\,(D_{n+1})^{m+2}\,\Gamma|\;|(D_{n+1})^{m+1}\Gamma|\,\varphi_k^2 \\ &\le C\left(\|\nabla_y\,(D_{n+1})^{m+2}\,\Gamma\,\varphi_k\|_2 + \|\,(D_{n+1})^{m+3}\,\Gamma\,\varphi_k\|_2\right)\|\,(D_{n+1})^{m+1}\,\Gamma\,\varphi_k\|_2 \\ &\le CC_1^{m^2}\left((2^k\ell(Q))^{-(m+1)}a_{m+1} + C_1^{(m+2)^2}(2^k\ell(Q))^{-(m+2)-\frac{n}{2}}\right)(2^k\ell(Q))^{-m-\frac{n}{2}} \end{aligned}$$

(where we have used (2.3))

$$\begin{aligned} &\le CC_1^{m^2}\left(a_{m+1}(2^k\ell(Q))^{-2m-1-\frac{n}{2}} + C_1^{(m+2)^2}(2^k\ell(Q))^{-2m-2-n}\right) \\ &\le C\delta a_{m+1}^2(2^k\ell(Q))^{-2m} + CC_1^{m^2+(m+2)^2}\left(\delta^{-1}+1\right)(2^k\ell(Q))^{-2m-2-n}, \end{aligned}$$

where $\delta > 0$ is at our disposal. Also, after integrating by parts

$$\begin{aligned} I'_{m,1} &= -C\Re e\sum_{i=1}^n\int A^*_{i,n+1}\,(\partial_t)^{m+2}\Gamma(x,t,y,0)\;(\partial_t)^{m+1}\,\overline{D_i\Gamma(x,t,y,0)}\,\varphi_k^2(y)dy \\ &\quad - C\Re e\sum_{i=1}^n\int A^*_{i,n+1}\,(\partial_t)^{m+2}\Gamma(x,t,y,0)\;(\partial_t)^{m+1}\,\overline{\Gamma(x,t,y,0)}\,D_i\varphi_k^2dy. \end{aligned}$$

By Cauchy's inequality, (2.3) and the bound for $\|\nabla\varphi_k\|_\infty$, we obtain

$$|I'_{m,1}| \le C\delta I_m + CC_1^{2(m+1)^2}\left(\delta^{-1}+1\right)(2^kQ)^{-2m-n-2}.$$

Similarly,

$$|I''_m| \le C\delta I_m + CC_1^{2m^2}\delta^{-1}(2^k\ell(Q))^{-2m-n-2}.$$

Collecting our estimates for $I'_{m,1}$, $I'_{m,2}$, and I''_m, we obtain for δ small enough that

$$\left(2^k\ell(Q)\right)^{2m} I_m = a_m^2 \le C\delta a_{m+1}^2 + CC_1^{2(m+2)^2}\delta^{-1}(2^k\ell(Q))^{-n-2}.$$

Thus,

$$\sum_{m=-1}^{\infty} C_2^{-m^2} a_m^2 \leq \sum_{m=-1}^{\infty} C_2^{-m^2} C\delta a_{m+1}^2 + \sum_{m=-1}^{\infty} C_2^{-m^2} C C_1^{2(m+2)^2} \delta^{-1} (2^k \ell(Q))^{-n-2}.$$

We now choose $\delta = \delta_m = \frac{1}{2C} C_2^{-2m-1}$, so that the right side of the last inequality equals

$$\frac{1}{2} \sum_{m=-1}^{\infty} C_2^{-(m+1)^2} a_{m+1}^2 + 2C \sum_{m=-1}^{\infty} C_2^{-m^2+2m+1} C_1^{2(m+2)^2} (2^k \ell(Q))^{-n-2}.$$

Choosing now $C_2 = C_1^3$, we obtain (2.6), under the a priori assumption that

$$\sum_{m=-1}^{\infty} C_2^{-m^2} a_m^2 < \infty.$$

The latter holds if $A(x) \in C^\infty$, for in that case $(\partial_t)^{m+1} \nabla_y \Gamma(x, t, y)$ satisfies point-wise bounds analogous to (2.3), possibly depending on the regularization of the coefficients. The constants in (2.6) and in the conclusion of Lemma 2.5 are independent of this regularization.

The proof of estimate (ii) is similar, except that we replace the cut-off function φ_k by $\varphi \in C_0^\infty(3Q)$, with $\varphi \equiv 1$ on $2Q$. We omit the details.

To finish the proof of the lemma, it remains to remove the a priori assumption of smoothness of the coefficients. To this end, fix a cube Q, and let $g \in C_0^\infty(Q)$, $\vec{f} \in C_0^\infty(R_k(Q), \mathbb{C}^n)$, where $R_0(Q) \equiv 2Q$, and $R_k(Q) \equiv 2^{k+1}Q \setminus 2^k Q$, $k \in \mathbb{N}$. It is enough to prove the estimate

$$|\langle g, (D_{n+1})^{m+1} S_t\, (\mathrm{div}_\parallel\, \vec{f})\rangle| \leq C C_2^{m^2/2} (2^k \ell(Q))^{-\frac{n}{2}-m-1} \|g\|_1 \|\vec{f}\|_2,$$

with $t > 0$, and, when $k = 0$, $\rho^{-1}\ell(Q) \leq t \leq \rho\ell(Q)$, with the constants depending upon ρ in the latter situation. The case $t < 0$ may be handled by an identical argument, which we omit. We define

$$A_\varepsilon \equiv P_\varepsilon A \equiv \phi_\varepsilon * A,$$

where $\phi_\varepsilon(x) \equiv \varepsilon^{-n}\phi(x/\varepsilon)$, and $\phi \in C_0^\infty(\{|x| < 1\})$ is non-negative and even, with $\int_{\mathbb{R}^n} \phi = 1$. Then $A_\varepsilon \to A$ a.e.. Set

$$L_\varepsilon \equiv -\,\mathrm{div}\, A_\varepsilon \nabla,$$

and let Γ_ε denote the corresponding fundamental solution. We note that

$$L_\varepsilon^{-1} - L^{-1} = L_\varepsilon^{-1} L\, L^{-1} - L_\varepsilon^{-1} L_\varepsilon L^{-1} = L_\varepsilon^{-1} div(A_\varepsilon - A)\nabla L^{-1}.$$

We choose a non-negative even cut-off function $\varphi \in C_0^\infty(-1, 1)$, with $\int_{\mathbb{R}} \varphi = 1$. Fix $t > 0$ (or $t \in (\rho^{-1}\ell(Q), \rho\ell(Q))$ if $k = 0$). For $\delta > 0$, set $\varphi_\delta(s) \equiv \delta^{-1}\varphi(s/\delta)$, and define

$$\vec{f}_\delta(y, s) \equiv \vec{f}(y)\, \varphi_\delta(s), \qquad g_{t,\delta}(x, \tau) \equiv g(x)\, \varphi_\delta(t - \tau)$$

Now, fix $\varepsilon > 0$ and suppose that $0 < \delta < t/8$. Then for $|t - \tau| < \delta$, we have

$$\begin{aligned}
(D_{n+1})^{m+1} L_\varepsilon^{-1}\, \mathrm{div}_\parallel\, \vec{f}_\delta(x, \tau) &= \iint (\partial_\tau)^{m+1} \Gamma_\varepsilon(x, \tau, y, s)\, \mathrm{div}_\parallel\, \vec{f}(y)\, \varphi_\delta(s)\, dyds \\
&= \int \varphi_\delta(s)\, (D_{n+1})^{m+1} \left(S_{\tau-s}^{L_\varepsilon}\, \mathrm{div}_\parallel\, \vec{f} \right)(x)\, ds,
\end{aligned}$$

where $S_t^{L_\varepsilon}$ denotes the single layer potential operator associated to L_ε. Thus,

$$\begin{aligned}
\left| \langle g_{t,\delta}, (D_{n+1})^{m+1} L_\varepsilon^{-1}\, \mathrm{div}_\parallel\, \vec{f}_\delta \rangle \right| &= \left| \iint \varphi_\delta(\tau)\varphi_\delta(s)\, \langle g, (D_{n+1})^{m+1} \left(S_{t-(\tau+s)}^{L_\varepsilon} \nabla_\parallel \right) \cdot \vec{f} \rangle\, dsd\tau \right| \\
&\leq C C_2^{m^2/2}\, \|g\|_1 \|\vec{f}\|_2\, (2^k \ell(Q))^{-\frac{n}{2}-m-1}
\end{aligned}$$

by the a priori bound obtained for smooth coefficients, since $|\tau+s| < 2\delta \le t/4$ and $\|\varphi\|_1 = 1$. Moreover,

$$\begin{aligned}|\langle g_{t,\delta}, (D_{n+1})^{m+1}\Big(L_\varepsilon^{-1} - L^{-1}\Big) \operatorname{div}_\| \vec{f_\delta}\rangle| &= \langle (D_{n+1})^{m+1} g_{t,\delta}, L_\varepsilon^{-1} \operatorname{div}(A_\varepsilon - A)\nabla L^{-1} \operatorname{div}_\| \vec{f_\delta}\rangle| \\ &= \langle \nabla (L_\varepsilon^*)^{-1} (D_{n+1})^{m+1} g_{t,\delta}, (A_\varepsilon - A)\nabla L^{-1} \operatorname{div}_\| \vec{f_\delta}\rangle|,\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$, for each fixed $\delta > 0$, by dominated convergence, since

$$\nabla (L_\varepsilon^*)^{-1} (D_{n+1})^{m+1} g_{t,\delta},\ \ \nabla L^{-1} \operatorname{div}_\| \vec{f_\delta} \in L^2(\mathbb{R}^{n+1}).$$

(For the first term, the case $m = -1$ uses that $C_0^\infty \subset L^{2_*} \hookrightarrow L^2_{-1}$, where $2_* = 2(n+1)/(n+3)$ is the lower Sobolev exponent in $n+1 \ge 3$ dimensions.) Thus,

$$|\langle g_{t,\delta}, (D_{n+1})^{m+1} L^{-1} \operatorname{div}_\| \vec{f_\delta}\rangle| \le C C_2^{m^2/2} \|g\|_1 \|\vec{f}\|_2 \, (2^k \ell(Q))^{-\frac{n}{2}-m-1}.$$

The conclusion of the lemma now follows from the observation that

$$\begin{aligned}\langle g_{t,\delta}, (D_{n+1})^{m+1} L^{-1} \operatorname{div}_\| \vec{f_\delta}\rangle &= \iint \varphi_\delta(\tau)\varphi_\delta(s) \langle g, (D_{n+1})^{m+1} S_{t-(\tau+s)} \operatorname{div}_\| \vec{f}\rangle \, ds d\tau \\ &\to \langle g, (D_{n+1})^{m+1} S_t \operatorname{div}_\| \vec{f}\rangle,\end{aligned}$$

as $\delta \to 0$, since $h(t) \equiv \langle g, (D_{n+1})^{m+1} S_t \operatorname{div}_\| \vec{f}\rangle$ is continuous (even C^∞) in $(0, \infty)$. □

As a Corollary of the previous two Lemmata we deduce

Lemma 2.7. *Suppose that L, L^* satisfy the standard assumptions, and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$. Then for every cube Q and for all integers $k \ge 1$ and $m \ge -1$, we have*

(i) $\|\partial_t^{m+1}(S_t\nabla) \cdot (\mathbf{f} 1_{2^{k+1}Q\setminus 2^k Q})\|^2_{L^2(Q)} \le C C_2^{m^2} 2^{-nk} (2^k \ell(Q))^{-2m-2} \|\mathbf{f}\|^2_{L^2(2^{k+1}Q\setminus 2^k Q)}, \quad t \in \mathbb{R}$

(ii) $\|\partial_t^{m+1}(S_t\nabla) \cdot (\mathbf{f} 1_{2Q})\|^2_{L^2(Q)} \le C_\rho^{m^2+1} \ell(Q)^{-2m-2} \|\mathbf{f}\|^2_{L^2(2Q)}, \quad \frac{\ell(Q)}{\rho} < |t| < \rho\, \ell(Q).$

Proof. We consider estimate (i). Let $x \in Q$. Then

$$\begin{aligned}|\partial_t^{m+1}(S_t\nabla) \cdot (\mathbf{f} 1_{2^{k+1}Q\setminus 2^k Q})(x)|^2 &= \left| \int_{2^{k+1}Q\setminus 2^k Q} \partial_t^{m+1} \nabla_{y,s}\Gamma(x,t,y,s)\mid_{s=0} \cdot \mathbf{f}(y) dy \right|^2 \\ &\le \|\partial_t^{m+1} \nabla_{y,s}\Gamma(x,t,y,s)\mid_{s=0}\|^2_{L^2(2^{k+1}Q\setminus 2^k Q)} \|\mathbf{f}\|^2_{L^2(2^{k+1}Q\setminus 2^k Q)} \\ &\le C C_2^{m^2} \left(2^k \ell(Q)\right)^{-n-2m-2} \|\mathbf{f}\|^2_{L^2(2^{k+1}Q\setminus 2^k Q)},\end{aligned}$$

where in the last step we have used Lemma 2.5(i) and (2.3). The bound (i) now follows from an integration over Q. The proof of (ii) is similar, and is omitted. □

Lemma 2.8. *Suppose that L, L^* satisfy the standard assumptions, and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$, $f : \mathbb{R}^n \to \mathbb{C}$. Then for every $t \in \mathbb{R}$, and for every integer $m \ge 0$, we have*

(i) $\|t^{m+1}\partial_t^{m+1}(S_t\nabla) \cdot \mathbf{f}\|_{L^2(\mathbb{R}^n)} \le C C_2^{m^2/2} \|\mathbf{f}\|_2$

(ii) $\|t^{m+1}\partial_t^{m+1}\nabla S_t f\|_{L^2(\mathbb{R}^n)} \le C C_2^{m^2/2} \|f\|_2.$

Proof. Fix $t \in \mathbb{R}$ and $m \ge 0$. It is enough to prove (i), since (ii) follows by duality and the fact that $adj\, S_t = S^*_{-t}$, where S^*_t is the single layer potential operator associated to L^*. We may further suppose that $t \ne 0$, since otherwise the left hand side of the inequality vanishes. Set $\theta_t = t^{m+1}\partial_t^{m+1}(S_t\nabla)$. We write

$$\|\theta_t \mathbf{f}\|_{L^2} = \left(\sum_Q \int_Q |\theta_t \mathbf{f}|^2 \right)^{1/2} = \left(\sum_Q \fint_Q \int_Q |\theta_t \mathbf{f}|^2 \right)^{1/2},$$

where the sum runs over the dyadic grid of cubes with $\ell(Q) \approx |t|$. With Q fixed, we decompose $\mathbf{f}$ into $\mathbf{f}1_{2Q}$ plus a sum of dyadic "annular" pieces $(\mathbf{f}1_{2^{k+1}Q\setminus 2^kQ})$. The bound (i) now follows from Lemma 2.7. We omit the details. $\square$

The next lemma says that

$$L = L_{\|} - \sum_{j=1}^{n+1} A_{n+1,j}\, D_{n+1} D_j - \sum_{i=1}^{n} D_i A_{i,n+1} D_{n+1}$$

in an appropriate weak sense on each "horizontal" cross-section.

Lemma 2.9. *Let L satisfy the standard assumptions of this paper. Suppose that $Lu = g$ in the strip $a < t < b$, where $g \in C_0^\infty(\mathbb{R}^{n+1})$. Suppose also that $\nabla u, \nabla\partial_t u \in L^2(\mathbb{R}^n)$, uniformly in $t \in (a, b)$, with norms depending continuously on $t \in (a, b)$. Then for every $F \in L^2(\mathbb{R}^n) \cap \dot{L}^2_1(\mathbb{R}^n)$, and for all $t \in (a, b)$, we have that*

$$\begin{aligned}\int_{\mathbb{R}^n} A_{\|}(x)\nabla_x\, u(x,t)\, \nabla_x F(x)\, dx \;=\; &\sum_{j=1}^{n+1} \int_{\mathbb{R}^n} A_{n+1,j}\,(x)\, \partial_{x_j}\partial_t\, u(x,t)\, F(x)\, dx \\ &- \sum_{i=1}^{n} \int_{\mathbb{R}^n} A_{i,n+1}(x)\, \partial_t u(x,t)\, \partial_{x_i} F(x)\, dx \;+\; \int_{\mathbb{R}^n} g(x,t)\, F(x) dx.\end{aligned} \tag{2.10}$$

Proof. Let $t \in (a, b)$, and let $\eta < \min(t - a, b - t)$. Set $\varphi_\eta(s) = \eta^{-1}\varphi(s/\eta)$, where $\varphi \in C_0^\infty(\frac{-1}{2}, \frac{1}{2})$, $0 \leq \varphi$, $\int \varphi = 1$. Define

$$F_{t,\eta}(x, s) \equiv F(x)\varphi_\eta(t - s).$$

Then by the definition of weak solutions, and t-independence, we have

$$\begin{aligned}\iint A_{\|}(x)\nabla_x\, u(x,s)\, \nabla_x F_{t,\eta}(x,s)\, dxds \;=\; &\sum_{j=1}^{n+1} \iint A_{n+1,j}\,(x)\, \partial_{x_j}\partial_t\, u(x,s)\, F_{t,\eta}(x,s)\, dxds \\ - \sum_{i=1}^{n} \iint A_{i,n+1}(x)\, \partial_s u(x,s)\, &\partial_{x_i} F_{t,\eta}(x,s)\, dxds \;+\; \iint g(x,s)\, F_{t,\eta}(x,s) dxds.\end{aligned}$$

By our hypotheses, the functions of t defined by the four integrals in (2.10), are all continuous in (a, b). The conclusion of the lemma then follows if we let $\eta \to 0$. $\square$

We may now prove a "2-sided" version of Lemma 2.8.

Lemma 2.11. *Suppose that L, L^* satisfy the standard assumptions, and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$. Then for every $t \in \mathbb{R}$, and for every integer $m \geq 0$, we have*

$$\|t^{m+2}\nabla_{\|}\partial_t^{m+1}\,(S_t\nabla) \cdot \mathbf{f}\|_2 \leq C_m \|\mathbf{f}\|_2.$$

Proof. Fix $t \in \mathbb{R}$. We may suppose that $t \neq 0$, since otherwise the left hand side vanishes. By Lemma 2.8 (ii) and $t-$independence, we may replace $(S_t\nabla)\cdot\mathbf{f}$ by $(S_t\nabla_{\|})\cdot \vec{f} = -S_t \operatorname{div}_{\|} \vec{f}$, where $\vec{f} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$. It then follows from Lemma 2.8 (ii) that

$$\beta_m(t) \equiv \|t^{m+2}\nabla_{\|}\partial_t^{m+1}\,(S_t\nabla_{\|}) \cdot \vec{f}\|_2^2 \leq C t^2 C_2^{m^2} \|\operatorname{div}_{\|} \vec{f}\|_2^2. \tag{2.12}$$

This last bound will not appear in our final quantitative estimates. Rather, the point is that the left hand side is *a priori* finite with some (non-optimal) quantitative control.

By ellipticity, Lemma 2.9 and Lemma 2.8 (i), we have that

$$\begin{aligned}
\beta_m(t) &\le Ct^{2m+4}\,\Re e\,\langle A_\parallel \nabla_\parallel \partial_t^{m+1} S_t \operatorname{div}_\parallel \vec{f}, \nabla_\parallel \partial_t^{m+1} S_t \operatorname{div}_\parallel \vec{f}\rangle \\
&= C\Re e \sum_{j=1}^{n+1} \langle A_{n+1,j}\, t^{m+3} D_j \partial_t^{m+2} S_t \operatorname{div}_\parallel \vec{f}, t^{m+1}\partial_t^{m+1} S_t \operatorname{div}_\parallel \vec{f}\rangle \\
&\qquad - C\Re e \sum_{i=1}^{n} \langle A_{i,n+1}\, t^{m+2} \partial_t^{m+2} S_t \operatorname{div}_\parallel \vec{f}, t^{m+2} D_i \partial_t^{m+1} S_t \operatorname{div}_\parallel \vec{f}\rangle \\
&\le C\delta^{-1} C_2^{m^2} \|\vec{f}\|_2^2 + C\delta C_2^{(m+2)^2} \|\vec{f}\|_2^2 + C\delta \beta_{m+1}(t) + C\delta^{-1} C_2^{(m+1)^2} \|\vec{f}\|_2 + C\delta \beta_m(t),
\end{aligned}$$

where δ is at our disposal. Choosing δ small enough, we may hide the last term, so that

$$\beta_m(t) \le C\delta^{-1} C_2^{(m+2)^2} \|\vec{f}\|_2^2 + C\delta \beta_{m+1}(t).$$

Thus, taking $\delta = \delta_m = \delta_0 C_2^{-2m}$, with δ_0 small, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} C_2^{-3m} C_2^{-(m+2)^2} \beta_m(t) &\le C \sum_{m=0}^{\infty} C_2^{-3m} \left(\delta_m^{-1} \|\vec{f}\|_2^2 + C_2^{-(m+2)^2} \delta_m \beta_{m+1}(t)\right) \\
&\le C \sum_{m=0}^{\infty} \left(\delta_0^{-1} C_2^{-m} \|\vec{f}\|_2^2 + \delta_0 C_2^{-3(m+1)} C_2^{-(m+3)^2} \beta_{m+1}(t)\right) \\
&\le C\|\vec{f}\|_2^2 + \frac{1}{2} \sum_{m=1}^{\infty} C_2^{-3m} C_2^{-(m+2)^2} \beta_m(t),
\end{aligned}$$

by choice of δ_0 small enough. By (2.12), the series converges, so the last term may be hidden on the left side of the inequality. In particular, we conclude that

$$\beta_m(t) \le C C_2^{(m+2)^2 + 3m} \|\vec{f}\|_2^2.$$

□

Lemma 2.13. *Suppose that L, L^* satisfy the standard assumptions. Fix a cube $Q \subset \mathbb{R}^n$, and suppose that $y, y' \in Q$. For $(x,t) \in \mathbb{R}^{n+1}$, set*

$$u(x,t) \equiv \Gamma(x,t,y,0) - \Gamma(x,t,y',0).$$

If α is the Hölder exponent in (2.4)*, then for every integer $k \ge 4$, we have*

$$\int_{2^{k+1}Q\setminus 2^k Q} |\nabla u(x,t)|^2 dx \le C 2^{-k\alpha} \left(2^k \ell(Q)\right)^{-n}. \tag{2.14}$$

Proof. By (2.4), it is enough to prove (2.14) with ∇_x in place of ∇. Let $\varphi_k \in C_0^\infty(3\cdot 2^k Q \setminus 3\cdot 2^{k-2}Q)$, with $\varphi_k \equiv 1$ on $2^{k+1}Q \setminus 2^k Q$ and $\|\nabla_x \varphi_k\|_\infty \le C(2^k \ell(Q))^{-1}$. Then the left hand side of (2.14) is bounded by an acceptable term involving a t derivative, plus

$$\begin{aligned}
\int |\nabla_x u(x,t)|^2 (\varphi_k(x))^2\, dx &\le C\Re e \int A_\parallel \nabla_x u \cdot \overline{\nabla_x u} \varphi_k^2 \\
&= C\Re e \int A_\parallel \nabla_x u \cdot \nabla_x \left(\overline{u} \varphi_k^2\right) - C\Re e \int A_\parallel \nabla_x u \cdot \nabla_x \left(\varphi_k^2\right) \overline{u} \equiv I + II.
\end{aligned}$$

By Lemma 2.2, for $y, y' \in Q$ and $x \in (2^{k-1}Q)^c$, we have

$$|u(x,t)| \le C 2^{-k\alpha} \left(2^k \ell(Q)\right)^{1-n}. \tag{2.15}$$

and also

$$|\partial_t u(x,t)| \le C 2^{-k\alpha} \left(2^k \ell(Q)\right)^{-n}. \tag{2.16}$$

Using the first of these, we obtain

$$|II| \leq C2^{-k\alpha}\left(2^k\ell(Q)\right)^{1-n}\|\nabla_x\varphi_k\|_\infty\int|\nabla_x u|\,\varphi_k$$
$$\leq C2^{-k\alpha}\left(2^k\ell(Q)\right)^{-n/2}\left(\int|\nabla_x u|^2\varphi_k^2\right)^{1/2} \leq C\varepsilon^{-1}2^{-2k\alpha}\left(2^k\ell(Q)\right)^{-n}+\varepsilon\int|\nabla_x u|^2\varphi_k^2,$$

where ε is at our disposal. Moreover, by Lemma 2.9,

$$I = -C\,\Re e\sum_{i=1}^{n}\left\{\int A_{i,n+1}\partial_t u\,\overline{D_i u}\varphi_k^2+\int A_{i,n+1}\partial_t u\,D_i\left(\varphi_k^2\right)\overline{u}\right\}$$
$$+\,C\,\Re e\sum_{j=1}^{n+1}\int A_{n+1,j}\,D_j\partial_t u\,\overline{u}\,\varphi_k^2 \equiv I_1+I_2+I_3.$$

Now, I_1 satisfies exactly the same bound as term II, and by essentially the same argument, except that we use (2.16) in place of (2.15). Moreover, using (2.15), (2.16), and the properties of φ_k, we see that

$$|I_2| \leq C2^{-2k\alpha}\left(2^k\ell(Q)\right)^{-n}.$$

To handle I_3, we note first that the case $m=0$ of Lemma 2.5(i) (with the roles of x and y reversed), applied separately for y and y', implies that

$$\int|\partial_t\nabla_x u(x,t)|^2\varphi_k^2 dx \leq C\left(2^k\ell(Q)\right)^{-n-2}.$$

Thus, using also (2.15), we have

$$|I_3| \leq C2^{-k\alpha}\left(2^k\ell(Q)\right)^{-n}.$$

Collecting these estimates, choosing ε sufficiently small, and hiding the small term on the left hand side of the inequality, we obtain the desired bound. □

In the sequel, we shall find it useful to consider approximations of the single layer potential. The bounds in the following lemma will not be used quantitatively, but will serve rather to justify certain formal manipulations. For $\eta>0$, set

$$S_t^\eta \equiv \int_{\mathbb{R}}\varphi_\eta(t-s)\,S_s\,ds, \tag{2.17}$$

where $\varphi_\eta \equiv \tilde{\varphi}_\eta * \tilde{\varphi}_\eta$, $\tilde{\varphi}_\eta \in C_0^\infty(-\eta/2,\eta/2)$ is non-negative and even, with $\int\tilde{\varphi}_\eta = 1$ and $\tilde{\varphi}_\eta(t)\equiv\eta^{-1}\tilde{\varphi}(t/\eta)$.

Lemma 2.18. *Suppose that L, L^* satisfy the standard assumptions, and let S_t denote the single layer potential operator associated to L. Then for each $\eta>0$ and for every $f\in L^2(\mathbb{R}^n)$ with compact support, we have*

(i) $\|\partial_t S_t^\eta f\|_{L^2(\mathbb{R}^n)} \leq C_{\beta,\eta}\|f\|_{L^{2n/(n+2\beta)}(\mathbb{R}^n)}, \quad 0<\beta<1.$
(ii) $\|\nabla_x S_t^\eta f\|_{L^2(\mathbb{R}^n)} \leq C_\eta\|f\|_{L^{(2n+2)/(n+3)}(\mathbb{R}^n)}$
(iii) $|||t\partial_t^2 S_t^\eta f||| \leq C_{\beta,\eta}\|f\|_{L^{2n/(n+2\beta)}(\mathbb{R}^n)}, \quad 0<\beta<1.$
(iv) $\|\nabla\left(S_t^\eta - S_t\right)f\|_{L^2(\mathbb{R}^n)} \leq C\frac{\eta}{|t|}\|f\|_2, \quad \eta<|t|/2.$
(v) $\lim_{\eta\to 0}\int_\varepsilon^\infty\int_{\mathbb{R}^n}|t\nabla\partial_t\left(S_t^\eta - S_t\right)f|^2\,\frac{dx\,dt}{t}=0, \quad 0<\varepsilon<1.$
(vi) *For each cube $Q\subset\mathbb{R}^n$,* $\|\partial_t S_t^\eta\|_{L^2(Q)\to L^2(\mathbb{R}^n)} \leq C_{\eta,\ell(Q)}$.

Proof. (i). We observe that

$$\partial_t S_t^\eta f(x) = \int_{\mathbb{R}^n} k_t(x,y) f(y)dy,$$

where $k_t(x,y) \equiv \partial_t \left(\varphi_\eta * \Gamma(x,\cdot,y,0)\right)(t)$. Thus, by Lemma 2.2

$$|k_t(x,y)| \leq C \min\left(|x-y|^{-n}, \eta^{-1}|x-y|^{1-n}\right) \;\leq\; C\eta^{-\beta}|x-y|^{\beta-n}, \quad 0<\beta<1.$$

Estimate (i) now follows by the fractional integral theorem.

(ii). We first note that

$$\begin{aligned} S_t^\eta f(x) &= \iiint \Gamma(x, t-s-\sigma, y, 0) f(y) dy \tilde{\varphi}_\eta(s)\tilde{\varphi}_\eta(\sigma) ds d\sigma \\ &= \int \left(L^{-1} f_\eta\right)(x, t-\sigma)\tilde{\varphi}_\eta(\sigma)d\sigma, \end{aligned}$$

where $f_\eta(y,s) \equiv f(y)\tilde{\varphi}_\eta(s)$. Let $\vec{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, with $\|\vec{g}\|_2 = 1$, and set $\vec{g}_\eta(x,\sigma) \equiv \vec{g}(x)\tilde{\varphi}_\eta(\sigma)$. Then

$$\begin{aligned} |\langle \vec{g}, \nabla_x S_t^\eta f\rangle| &= \left| \iint \operatorname{div}_x \vec{g}_\eta(x,\sigma)\overline{\left(L^{-1} f_\eta\right)}(x, t-\sigma) dx d\sigma \right| \\ &\leq \|\vec{g}_\eta\|_{L^2(\mathbb{R}^{n+1})}\|\nabla L^{-1} f_\eta\|_{L^2(\mathbb{R}^{n+1})} \;\leq\; C\eta^{-1/2}\|f_\eta\|_{L^{2_*}(\mathbb{R}^{n+1})} \equiv C\eta^{-1/2}\|\varphi_\eta\|_{L^{2_*}(\mathbb{R})}\|f\|_{L^{2_*}(\mathbb{R}^n)}, \end{aligned}$$

where $2_* = (2n+2)/(n+3)$, since $L^{2_*}(\mathbb{R}^{n+1}) \hookrightarrow \dot{L}^2_{-1}(\mathbb{R}^{n+1}) \equiv \left(\dot{L}^2_1(\mathbb{R}^{n+1})\right)^*$, and $\nabla L^{-1}\operatorname{div} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$. Estimate (ii) now follows.

(iii). We proceed as for estimate (i), and write

$$t\partial_t^2 S_t^\eta f(x) = \int_{\mathbb{R}^n} h_t(x,y) f(y)dy,$$

where $h_t(x,y) \equiv t\partial_t^2 \left(\varphi_\eta * \Gamma(x,\cdot,y,0)\right)(t)$, so that, by Lemma 2.2,

$$|h_t(x,y)| \leq Ct \min\left(|x-y|^{-n-1}, \eta^{-2}|x-y|^{1-n}\right) \;\leq\; Ct\,\eta^{-1-\beta}|x-y|^{\beta-n}, \quad 0<\beta<1.$$

Moreover, if $t > 2\eta$, we have the sharper estimate

$$|h_t(x,y)| \leq C\frac{t}{(t+|x-y|)^{n+1}} \leq Ct^{-\beta}|x-y|^{\beta-n}, \quad 0<\beta<1.$$

Thus,

$$|||t\partial_t^2 S_t^\eta f|||^2 \leq C\left(\int_0^{2\eta} \eta^{-2-2\beta} t dt + \int_{2\eta}^\infty t^{-1-2\beta}dt\right)\|f\|^2_{L^{2n/(n+2\beta)}(\mathbb{R}^n)},$$

and (iii) follows.

(iv). We suppose that $\eta < |t|/2$. Then

$$\|\nabla\left(S_t^\eta - S_t\right) f\|_{L^2(\mathbb{R}^n)} \leq \varphi_\eta * \|\nabla\left(S_{(\cdot)} - S_t\right) f\|_{L^2(\mathbb{R}^n)}.$$

But for $|s-t| < \eta < |t|/2$, we have by the mean value theorem and Lemma 2.8(ii) that

$$\|\nabla\left(S_s - S_t\right) f\|_{L^2(\mathbb{R}^n)} \leq \frac{\eta}{|t|} \sup_{|\tau-t|<|t|/2} \|\tau\nabla\partial_\tau S_\tau f\|_{L^2(\mathbb{R}^n)} \leq C\frac{\eta}{|t|}\|f\|_2.$$

(v). We take $\eta < \varepsilon/2$, and write

$$\begin{aligned}\int_\varepsilon^\infty \int_{\mathbb{R}^n} |t\nabla\partial_t \left(S_t^\eta - S_t\right) f|^2 \frac{dx\,dt}{t} &= \int_\varepsilon^\infty \int_{\mathbb{R}^n} |\varphi_\eta * t\nabla D_{n+1}\left(S_{(\cdot)} - S_t\right) f|^2 \frac{dx\,dt}{t} \\ &\leq \int_\varepsilon^\infty \varphi_\eta * \|t\nabla D_{n+1}\left(S_{(\cdot)} - S_t\right) f\|_2^2 \frac{dt}{t} \qquad (2.19)\end{aligned}$$

We claim that the last expression converges to 0, as $\eta \to 0$. Indeed, for $|s-t| < \eta < t/2$, we have that

$$\|t\nabla D_{n+1}\left(S_s - S_t\right) f\|_{L^2(\mathbb{R}^n)} \leq \eta \sup_{|\tau - t| < t/2} \|\tau\nabla\partial_\tau^2 S_\tau f\|_{L^2(\mathbb{R}^n)} \leq C\frac{\eta}{t}\|f\|_2$$

by Lemma 2.8(ii). Thus, for $\eta < \varepsilon/2$, (2.19) is bounded by $C\eta^2\varepsilon^{-2}\|f\|_2^2$, and the claim follows.

(vi). Estimate (vi) follows from (i) and Hölder's inequality. □

3. S “ - ”

Here, we prove some estimates that hold in general for operators satisfying the conclusions of Lemmas 2.7 and 2.8. For the sake of notational convenience, we observe that part (i) of the former conclusion can be reformulated as

$$\|\theta_t(f 1_{2^{k+1}Q\setminus 2^kQ})\|_{L^2(Q)}^2 \leq C_m 2^{-nk}\left(\frac{|t|}{2^k\ell(Q)}\right)^{2m+2} \|f\|_{L^2(2^{k+1}Q\setminus 2^kQ)}^2 \qquad (3.1)$$

where $\theta_t = t^{m+1}\partial_t^{m+1}(S_t\nabla)$. We now consider generic operators θ_t which satisfy (3.1) for some integer $m \geq 0$. The next lemma is essentially due to Fefferman and Stein [FS]. We omit the well known proof.

Lemma 3.2. *Suppose that $\{\theta_t\}_{t\in\mathbb{R}}$ is a family of operators which satisfies* (3.1) *for some integer $m \geq 0$ and in every cube Q, whenever $|t| \leq C\ell(Q)$. If $|||\theta_t|||_{op} \leq C$, then*

$$|\theta_t b(x)|^2 \frac{dxdt}{|t|}$$

is a Carleson measure for every $b \in L^\infty$.

Lemma 3.3. *Suppose that $\{\theta_t\}_{t\in\mathbb{R}}$ is a family of operators satisfying* (3.1) *for some integer $m \geq 0$, as well as the bound*

$$\sup_{t\in\mathbb{R}} \|\theta_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_2.$$

Suppose that $\{\Lambda_t\}_{t\in\mathbb{R}}$ is a family of operators satisfying the bounds

$$\sup_{t\in\mathbb{R}} \|\Lambda_t f\|_2 \leq C\|f\|_2, \qquad \|\Lambda_t f\|_{L^2(E)} \leq C\exp\left\{\frac{-\operatorname{dist}(E,E')}{C|t|}\right\}\|f\|_{L^2(E')}$$

whenever (in the latter estimate) support $f \subseteq E'$. Then $\theta_t\Lambda_t$ also satisfies (3.1)*, whenever $|t| \leq C\ell(Q)$.*

Proof. We may suppose that $k \geq 4$, otherwise, subdivide Q dyadically to reduce to this case. Given Q, set $\tilde{Q} \equiv 2^{k-2}Q$. Then

$$\theta_t\Lambda_t = \theta_t 1_{\tilde{Q}}\Lambda_t + \theta_t 1_{\mathbb{R}^n\setminus\tilde{Q}}\Lambda_t. \qquad (3.4)$$

For the first term, we have the bound

$$\begin{aligned}\|\theta_t 1_{\tilde{Q}}\Lambda_t\left(f 1_{2^{k+1}Q\setminus 2^kQ}\right)\|_{L^2(Q)} &\leq \|\theta_t\|_{2\to2}\|\Lambda_t\left(f 1_{2^{k+1}Q\setminus 2^kQ}\right)\|_{L^2(\tilde{Q})} \\ &\leq \|\theta_t\|_{2\to2}\exp\left\{\frac{-2^k\ell(Q)}{C|t|}\right\}\|f\|_{L^2(2^{k+1}Q\setminus 2^kQ)}\end{aligned}$$

which in particular yields (3.1) for this term, if $|t| \leq C\ell(Q)$. Next, we consider the second term in (3.4), which equals

$$\sum_{j\geq k-2} \theta_t 1_{2^{j+1}Q\setminus 2^j Q}\Lambda_t.$$

The desired bound follows for this term by applying (3.1) for each j fixed, and summing the resulting geometric series. □

Lemma 3.5. (i). *Suppose that* $\{R_t\}_{t\in\mathbb{R}}$ *is a family of operators satisfying* (3.1)*, for some* $m \geq 1$*, and for all* $|t| \leq C\ell(Q)$*, and suppose also that* $\sup_t \|R_t\|_{2\to 2} \leq C$*, and that* $R_t 1 = 0$ *for all* $t \in \mathbb{R}$ *(our hypotheses allow* $R_t 1$ *to be defined as an element of* L^2_{loc}*). Then for* $h \in \dot{L}^2_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |R_t h|^2 \leq Ct^2 \int_{\mathbb{R}^n} |\nabla_x h|^2. \tag{3.6}$$

(ii). *If, in addition,* $\|R_t \operatorname{div}_x\|_{2\to 2} \leq C/|t|$*, then also*

$$|||R_t f||| \leq C\|f\|_2. \tag{3.7}$$

Proof. We suppose that $t > 0$, and show that (3.6) implies (3.7). The latter follows from

$$\|R_t\left(s^2\Delta\, e^{s^2\Delta}\right)\|_{2\to 2} \leq C\,\min\left(\frac{s}{t}, \frac{t}{s}\right), \tag{3.8}$$

by a standard orthogonality argument. In turn, (3.8) is easy to prove: the case $t < s$ is just (3.6), and the case $s < t$ follows by hypothesis from the factorization $\Delta = \operatorname{div}_x \nabla_x$.

We now turn to the proof of (3.6). Let $\mathbb{D}(t)$ denote the grid of dyadic cubes with $\ell(Q) \leq |t| < 2\ell(Q)$. For convenience of notation we set $m_Q h \equiv \fint_Q h$. Then

$$\left(\int_{\mathbb{R}^n} |R_t h|^2\right)^{\frac{1}{2}} = \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |R_t h|^2\right)^{\frac{1}{2}} = \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |R_t(h - m_{2Q}h)|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |R_t[(h - m_{2Q}h)1_{2Q}]|^2\right)^{\frac{1}{2}} + \left(\sum_{Q\in\mathbb{D}(t)} \int_Q |R_t[(h - m_{2Q}h)1_{(2Q)^c}]|^2\right)^{\frac{1}{2}} \equiv I + II.$$

Since $R_t : L^2 \to L^2$, we have by Poincaré's inequality that

$$I \leq C\left(\sum_{Q\in\mathbb{D}(t)} \int_{2Q} |h - m_{2Q}h|^2\right)^{\frac{1}{2}} \leq C|t|\left(\sum_{Q\in\mathbb{D}(t)} \int_{2Q} |\nabla_x h|^2\right)^{\frac{1}{2}} \leq C|t|\,\|\nabla_x h\|_2.$$

Moreover, we are given that R_t satisfies (3.1). Thus,

$$II \leq \sum_{k=1}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)} \int_Q |R_t[(h - m_{2Q}h)1_{2^{k+1}Q\setminus 2^k Q}]|^2\right)^{\frac{1}{2}}$$

$$\leq C\sum_{k=1}^{\infty}\left(\sum_{Q\in\mathbb{D}(t)} 2^{-k(n+4)}\int_{2^{k+1}Q} |h - m_{2Q}h|^2\right)^{\frac{1}{2}}$$

$$\leq C\sum_{k=1}^{\infty}\sum_{j=1}^{k}\left(\sum_{Q\in\mathbb{D}(t)} 2^{-4k}2^{-jn}\int_{2^{j+1}Q} |h - m_{2^{j+1}Q}h|^2\right)^{\frac{1}{2}},$$

where in the last step we have used that

$$h - m_{2Q}h = h - m_{2^{k+1}Q}h + m_{2^{k+1}Q}h - m_{2^k Q}h + \cdots - \cdots + m_{4Q}h - m_{2Q}h.$$

By Poincaré's inequality, since $j \le k$ we obtain in turn the bound

$$C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} 2^{-jn} \int_{2^{j+1}Q} |\nabla_x h|^2 \right)^{\frac{1}{2}} \le C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left(\sum_{Q \in \mathbb{D}(t)} \int_Q \fint_{2^{j+1}Q} |\nabla_x h|^2 \right)^{\frac{1}{2}}$$
$$\le C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left(\int_{\mathbb{R}^n} \fint_{|x-y| \le C2^j t} |\nabla_x h(x)|^2 dxdy \right)^{\frac{1}{2}} = C|t| \, \|\nabla_x h\|_2.$$

□

Lemma 3.9. *Given* $\{R_t\}_{t \in \mathbb{R}_+}$ *as in part* **(i)** *of the previous lemma, we have that*

$$|||t^{-1} R_t F||| \le C \|\nabla_x F\|_{L^2(\mathbb{R}^n)},$$

provided that $\left| \frac{1}{t} R_t \Phi(x) \right|^2 \frac{dxdt}{|t|}$ *is a Carleson measure, where* $\Phi(x) \equiv x$.

Proof. We may assume that $F \in C_0^\infty$, and that $t > 0$. Let $\mathbb{D}_j$ denote the dyadic grid of cubes of side length 2^{-j}. Then

$$\begin{aligned} |||t^{-1} R_t F|||^2 &= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_{-j}} \int_{2^j}^{2^{j+1}} \int_Q |t^{-1} R_t F(y)|^2 dy \frac{dt}{t} \\ &= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_{-j}} \int_{2^j}^{2^{j+1}} \int_Q \fint_Q |t^{-1} R_t F(y)|^2 dydx \frac{dt}{t}. \end{aligned} \tag{3.10}$$

We now use an idea taken from [J] and [Ch2, pp. 32-33]. For (x, t) fixed, set

$$G_{x,t}(z) \equiv F(z) - F(x) - (z - x) \cdot P_t \left(\nabla_\| F \right)(x),$$

where as usual P_t is an approximate identity. Since $R_t 1 = 0$, we have, for any fixed x,

$$\frac{1}{t} R_t F(y) = \frac{1}{t} R_t \left(G_{x,t} \right)(y) + \frac{1}{t} R_t \Phi(y) \cdot P_t \left(\nabla_\| F \right)(x) \equiv I + II.$$

The contribution of II to (3.10) is bounded by

$$\sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_{-j}} \int_{2^j}^{2^{j+1}} \int_Q |P_t \left(\nabla_\| F \right)(x)|^2 \fint_Q \left| \frac{1}{t} R_t \Phi(y) \right|^2 dydx \frac{dt}{t}$$
$$\le C \int_0^\infty \int_{\mathbb{R}^n} |P_t \left(\nabla_\| F \right)(x)|^2 \left\{ \fint_{B(x,Ct)} \left| \frac{1}{t} R_t \Phi(y) \right|^2 dy \right\} dx \frac{dt}{t} \le C \|\nabla_\| F\|^2_{L^2(\mathbb{R}^n)} \|\mu\|_C,$$

by Carleson's Lemma, where

$$\|\mu\|_C \equiv \sup_Q \int_0^{\ell(Q)} \fint_Q \left\{ \fint_{B(x,Ct)} \left| \frac{1}{t} R_t \Phi(y) \right|^2 dy \right\} dx \frac{dt}{t}$$
$$\le C \sup_Q \int_0^{\ell(Q)} \fint_{CQ} \left| \frac{1}{t} R_t \Phi(y) \right|^2 \fint_{|x-y| \le Ct} dxdy \frac{dt}{t} \le C \sup_Q \int_0^{\ell(Q)} \fint_Q \left| \frac{1}{t} R_t \Phi \right|^2 \frac{dxdt}{t}.$$

Next we consider the contribution of I to (3.10). For $Q \in \mathbb{D}_{-j}$, and $x \in Q$, we have

$$I = R_t \left(t^{-1} G_{x,t} 1_{2Q} \right)(y) + \sum_{k=1}^{\infty} R_t \left(t^{-1} G_{x,t} 1_{2^{k+1}Q \setminus 2^k Q} \right)(y) \equiv I_0 + \sum_{k=1}^{\infty} I_k.$$

Since $R_t : L^2 \to L^2$, we obtain the bound

$$
\begin{aligned}
|||I_0|||^2 &\leq C \sum_{j=-\infty}^{\infty} \sum_{Q\in\mathbb{D}_{-j}} \int_{2^j}^{2^{j+1}} \int_Q \fint_{2Q} \frac{|G_{x,t}(y)|^2}{t^2} dydx \frac{dt}{t} \\
&\leq C \int_0^\infty \int_{\mathbb{R}^n} (\beta(x,t))^2 \frac{dxdt}{t} \ \leq\ C\|\nabla_{||}F\|_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

where $(\beta(x,t))^2 = \fint_{|x-y|<Ct} t^{-2}|G_{x,t}(y)|^2 dy$, and where the last step is a well known consequence of Plancherel's Theorem, see, e.g. [Ch2, pp. 32-33] or [H, pp. 249-250]. Furthermore, since R_t satisfies (3.1) for some $m \geq 1$, whenever $t \approx \ell(Q)$, we have that

$$
\begin{aligned}
C^{-1} \sum_{k=1}^{\infty} |||I_k||| &\leq \sum_{k=1}^{\infty} \left(\sum_{j=-\infty}^{\infty} \sum_{Q\in\mathbb{D}_j} \int_{2^j}^{2^{j+1}} \int_Q \frac{1}{t^n 2^{k(n+4)}} \int_{|x-y|\leq C2^k t} \frac{|G_{x,t}(y)|^2}{t^2} \frac{dydxdt}{t} \right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{\infty} 2^{-k} \left(\int_0^\infty \int_{\mathbb{R}^n} \fint_{|x-y|\leq C2^k t} \frac{|G_{x,t}(y)|^2}{(2^k t)^2} dydx \frac{dt}{t} \right)^{\frac{1}{2}} \equiv \sum_{k=1}^{\infty} 2^{-k} |||\beta_k|||,
\end{aligned}
$$

where, after making the change of variable $t \to t/2^k$,

$$
\beta_k(x,t) = \left(\fint_{|x-y|\leq Ct} \frac{|F(y) - F(x) - (y-x)\cdot P_{2^{-k}t}(\nabla_{||}F)(x)|^2}{t^2} dy \right)^{1/2}.
$$

We now claim that $|||\beta_k||| \leq C\sqrt{k}\|\nabla_{||}F\|_2$, from which the conclusion of the lemma trivially follows. By Plancherel's Theorem, the definiton of P_t and the change of variable $x - y = h$ we have

$$
|||\beta_k|||^2 = \int_0^\infty \fint_{|h|<Ct} \int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot h} - 1 - (ih\cdot\xi)\,\hat{\phi}(2^{-k}t\xi)|^2}{t^2|\xi|^2} |\xi|^2 |\hat{F}(\xi)|^2 d\xi dh \frac{dt}{t},
$$

where $\phi \in C_0^\infty\{|x| < 1\}$ and $\int \phi \equiv 1$. By the change of variable $h \to th$, we have

$$
|||\beta_k|||^2 = \int_0^\infty \fint_{|h|<C} \int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot ht} - 1 - (iht\cdot\xi)\,\hat{\phi}(2^{-k}t\xi)|^2}{t^2|\xi|^2} |\xi|^2 |\hat{F}(\xi)|^2 \frac{d\xi dh dt}{t}.
$$

Since $\hat{\phi} \in \mathcal{S}$ and $\hat{\phi}(0) = 1$, we have that

$$
\frac{|e^{it\xi\cdot h} - 1 - (ih\cdot t\xi)\,\hat{\phi}(2^{-k}t\xi)|}{t|\xi|} \leq C \min\left(t|\xi|, 1, \frac{2^k}{t|\xi|} \right).
$$

Indeed, if $t|\xi| \leq 1$, then

$$
\begin{aligned}
\frac{|e^{it\xi\cdot h} - 1 - (ih\cdot t\xi)\,\hat{\phi}(2^{-k}t\xi)|}{t|\xi|} &\leq \frac{|e^{it\xi\cdot h} - 1 - ih\cdot t\xi|}{t|\xi|} + \frac{|ih\cdot t\xi\,\left(1 - \hat{\phi}(2^{-k}t\xi)\right)|}{t|\xi|} \\
&\leq C(t|\xi| + 2^{-k}t|\xi|) \leq Ct|\xi|.
\end{aligned}
$$

On the other hand, if $t|\xi| > 1$, then

$$
\frac{|e^{it\xi\cdot h} - 1|}{t|\xi|} \leq \frac{2}{t|\xi|},
$$

and

$$
\frac{|(ih\cdot t\xi)\,\hat{\phi}(2^{-k}t\xi)|}{t|\xi|} \leq C|\hat{\varphi}(2^{-k}t\xi)| \leq \frac{C}{1 + 2^{-k}t|\xi|} \leq C\min\left(1, \frac{2^k}{t|\xi|}\right).
$$

We then obtain the bound $|||\beta_k|||^2 \leq Ck\|\nabla_{||}F\|_2^2$ as claimed. $\square$

Lemma 3.11. *Suppose that θ_t satisfies* (3.1) *for some $m \geq 0$, whenever $0 < t \leq C\ell(Q)$ and that $\|\theta_t\|_{2\to 2} \leq C$. Let $b \in L^\infty(\mathbb{R}^n)$, and let $\mathcal{A}_t$ denote a self-adjoint averaging operator whose kernel $\varphi_t(x, y)$ satisfies $|\varphi_t(x, y)| \leq Ct^{-n}\, 1_{\{|x-y|<Ct\}}$, $\varphi_t \geq 0$, $\int \varphi_t(x, y)dy = 1$. Then*

$$\sup_{t\in\mathbb{R}_+} \|(\theta_t b)\mathcal{A}_t f\|_2 \leq C\|b\|_\infty \|f\|_2.$$

Proof. Since we do not assume that $\theta_t : L^\infty \to L^\infty$, this requires a bit of an argument. Observe that

$$\|(\theta_t b)\mathcal{A}_t f\|_2^2 \ \leq\ \|f\|_2 \|\mathcal{A}_t(|\theta_t b|^2 \mathcal{A}_t f)\|_2 \ \leq\ \|f\|_2^2 \|\mathcal{K}_t(x, \cdot)\|_{L^1(\mathbb{R}^n)},$$

where $\mathcal{K}_t(x, y)$ is the kernel of the self-adjoint operator $f \to \mathcal{A}_t(|\theta_t b|^2 \mathcal{A}_t f)$, i.e.,

$$\mathcal{K}_t(x, y) = \int_{\mathbb{R}^n} \varphi_t(x, z)|\theta_t b(z)|^2 \varphi_t(z, y)dz.$$

Consequently,

$$\|\mathcal{K}_t(x, \cdot)\|_{L^1} = \int_{\mathbb{R}^n} \varphi_t(x, z)|\theta_t b(z)|^2 dz \leq Ct^{-n} \int_{|x-z|<Ct} |\theta_t b(z)|^2 dz.$$

Thus, by (3.1) and the fact that θ_t is bounded on L^2 uniformly in t, we have that

$$\|\mathcal{K}_t(x, \cdot)\|_{L^1}^{1/2} \leq C \left\{ \left(\fint_{Q(x,4Ct)} |b|^2 \right)^{1/2} + \sum_{k=2}^{\infty} 2^{-k} \left(\fint_{Q(x,2^{k+1}Ct)\setminus Q(x,2^k Ct)} |b|^2 \right)^{1/2} \right\} \leq C\|b\|_\infty,$$

where $Q(x, Rt)$ is the cube centered at x with side length Rt. This proves the lemma. □

Lemma 3.12. *Suppose that*

$$\Omega_t = \int_0^t \left(\frac{s}{t}\right)^\delta W_{t,s}\, \theta_s \frac{ds}{s},$$

for some $\delta > 0$, where $\sup_{t,s} \|W_{t,s}\|_{2\to 2} \leq C$. Then

$$|||\Omega_t|||_{op} \leq C|||\theta_s|||_{op}.$$

Proof. This is a standard Schur type argument. Indeed, if $|||G(x, t)||| \leq 1$, then

$$\begin{aligned}
\left| \int_0^\infty \int_{\mathbb{R}^n} G(x, t)\Omega_t f(x)dx \frac{dt}{t} \right| &= \left| \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} 1_{\{s<t\}} \left(\frac{s}{t}\right)^\delta G(x, t) W_{t,s}\theta_s f(x) dx \frac{dt}{t} \frac{ds}{s} \right| \\
&\leq \left(\int_0^\infty \int_{\mathbb{R}^n} |G(x, t)|^2 \int_0^t \left(\frac{s}{t}\right)^\delta \frac{ds}{s} \frac{dxdt}{t} \right)^{\frac{1}{2}} \left(C \int_0^\infty \int_{\mathbb{R}^n} |\theta_s f(x)|^2 \int_s^\infty \left(\frac{s}{t}\right)^\delta \frac{dt}{t} \frac{dxds}{s} \right)^{\frac{1}{2}} \\
&\leq C|||\theta_s f|||.
\end{aligned}$$

□

4. T , ,

We begin by proving a useful technical lemma.

Lemma 4.1. *Let L, L^* satisfy the standard assumptions. Suppose that $Lu = 0$ and that $\widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n)$. Then*

$$\sup_{t>0} \|\nabla u(\cdot, t)\|_2 \leq C\|\widetilde{N}_*(\nabla u)\|_2. \tag{4.2}$$

Proof. The desired bound for $\partial_t u$ follows readily from t-independence and (1.3). Thus, we need only consider $\nabla_x u$. Let $\vec{\psi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, with $\|\vec{\psi}\|_2 = 1$. For $t_0 > 0$ fixed, it will then be enough to establish the bound

$$\left| \int_{\mathbb{R}^n} u(\cdot, t_0) \operatorname{div}_x \vec{\psi} \right| \leq C \|\widetilde{N}_*(\nabla u)\|_2.$$

To this end, we write

$$\begin{aligned} \int_{\mathbb{R}^n} u(\cdot, t_0) \operatorname{div}_x \vec{\psi} \quad &= \quad \int_{\mathbb{R}^n} \left(u(x, t_0) - \fint_{t_0/2}^{t_0} u(x,t) dt \right) \operatorname{div}_x \vec{\psi}(x) dx \\ &\qquad + \int_{\mathbb{R}^n} \fint_{t_0/2}^{t_0} u(x,t) \operatorname{div}_x \vec{\psi}(x)\, dt\, dx \;\equiv\; I + II. \end{aligned}$$

We first observe that

$$|II| = \left| \int_{\mathbb{R}^n} \fint_{t_0/2}^{t_0} \left(\fint_{|x-y|<t} dy \right) \nabla_x u(x,t)\, \vec{\psi}(x)\, dt\, dx \right| \leq C \|\widetilde{N}_*(\nabla u)\|_2,$$

by Cauchy-Schwarz and Fubini's Theorem. Moreover,

$$\begin{aligned} |I| &= \left| \int_{\mathbb{R}^n} \fint_{t_0/2}^{t_0} \int_t^{t_0} \partial_s u(x,s)\, ds \operatorname{div}_x \vec{\psi}(x)\, dt\, dx \right| \\ &= \left| \fint_{t_0/2}^{t_0} \int_t^{t_0} \int_{\mathbb{R}^n} \nabla_x \partial_s u(x,s)\, \vec{\psi}(x)\, dx\, ds\, dt \right| \leq C\, t_0 \left(\fint_{t_0/2}^{t_0} \int_{\mathbb{R}^n} |\nabla \partial_s u(x,s)|^2 dx\, ds \right)^{1/2} \\ &\qquad \leq C \left(\fint_{t_0/2}^{t_0} \int_{\mathbb{R}^n} |\partial_s u(x,s)|^2 dx\, ds \right)^{1/2}, \end{aligned}$$

where in the last step we have split $\mathbb{R}^n$ into cubes of side length $\approx t_0$ and used Caccioppoli's inequality. The conclusion of the lemma follows since the bound already holds for $\partial_s u$. $\square$

We now discuss some trace results. The following lemma is the analogue of Theorem 3.1 of [KP]. We recall that $u \to f$ *n.t.* means that $\lim_{(y,t)\to(x,0)} u(y,t) = f(x)$, for *a.e.* $x \in \mathbb{R}^n$, where the limit runs over $(y,t) \in \gamma(x)$. As usual, P_ε will denote a self-adjoint approximate identity acting in $\mathbb{R}^n$. We shall denote by $W_c^{1,2}$ the subspace of compactly supported elements of the usual Sobolev space $W^{1,2}$.

Lemma 4.3. *Suppose that L, L^* satisfy the standard assumptions. If $Lu = 0$ in $\mathbb{R}^{n+1}_+$ and $\widetilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n)$, then there exists $f \in \dot{L}^2_1(\mathbb{R}^n)$ such that*

(i) *$\|\nabla_\| f\|_2 \leq C\|\widetilde{N}_*(\nabla u)\|_2$, and $u \to f$ n.t., with $|u(y,t) - f(x)| \leq Ct\widetilde{N}_*(\nabla u)(x)$ whenever $(y,t) \in \gamma(x)$.*
(ii) *$\nabla_\| u(\cdot, t) \to \nabla_\| f$ weakly in $L^2(\mathbb{R}^n)$ as $t \to 0$.*

If $Lu = 0$ in $\mathbb{R}^n \times (0, \rho)$, where $0 < \rho \leq \infty$, and $\sup_{0<t<\rho} \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$, then there exists $g \in L^2(\mathbb{R}^n)$ such that $g = \partial u/\partial \nu$ in the variational sense, i.e.,

(iii) *$\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \nabla \Phi\, dx\, dt = \int_{\mathbb{R}^n} g\, \Phi\, dx, \quad \forall\, \Phi \in W_c^{1,2}(\mathbb{R}^n \times (-\rho, \rho))$.*
(iv) *$\vec{N} \cdot A\nabla u(\cdot, t) \to g$ weakly in $L^2(\mathbb{R}^n)$ as $t \to 0$.*

(Here, $\vec{N} \equiv -e_{n+1}$ is the unit outer normal to $\mathbb{R}^{n+1}_+$).

Of course, the analogous results hold for the lower half space.

Proof. The existence of $f \in \dot{L}^2_1(\mathbb{R}^n)$ satisfying (i) may be obtained by following *mutatis mutandi* the corresponding argument in [KP] pp. 461-462.

(*ii*). We first establish convergence in the sense of distributions. Let $\vec{\psi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$. Then by (*i*),

$$\left|\int_{\mathbb{R}^n} (\nabla_{\|} u(\cdot,t) - \nabla_{\|} f)\,\vec{\psi}\right| = \left|\int_{\mathbb{R}^n} (u(\cdot,t) - f)\ \mathrm{div}_{\|}\,\vec{\psi}\right| \le Ct\,\|\widetilde{N}_*(\nabla u)\|_2 \|\,\mathrm{div}_{\|}\,\vec{\psi}\|_2 \to 0.$$

By the density of C_0^∞ in L^2, the weak convergence in L^2 then follows readily from (4.2).

(*iii*). We follow [KP], with some modifications owing to the unboundedness of our domain. We treat only the case $\rho = \infty$, and leave it to the reader to check the details in the case of finite ρ. Fix $0 < R < \infty$ and set $B_R = B(0, R) \equiv \{X \in \mathbb{R}^{n+1} : |X| < R\}$, $B_R^\pm \equiv B_R \cap \mathbb{R}^{n+1}_\pm$ and $\Delta_R = B_R \cap \{t = 0\}$. Define a linear functional on $W_0^{1,2}(B_R)$ (the closure of C_0^∞ in $W^{1,2}(B_R)$) by

$$\Lambda_R(\Psi) \equiv \iint_{B_R^+} A\nabla u \cdot \overline{\nabla\Psi},\ \ \Psi \in W_0^{1,2}(B_R).$$

Clearly, $\|\Lambda_R\| \le CR^{1/2}\sup_{t>0}\|\nabla u(\cdot,t)\|_2$. By trace theory, $tr\left(W_0^{1,2}(B_R)\right) \subset H_0^{1/2}(\Delta_R)$, defined as the closure in $H^{1/2}(\mathbb{R}^n)$ of $C_0^\infty(\Delta_R)$. Here, $\|f\|_{H^s(\mathbb{R}^n)} \equiv \|f\|_{L^2(\mathbb{R}^n)} + \||\xi|^s\hat{f}\|_{L^2(\mathbb{R}^n)}$, for $0 \le s \le 1$. On the other hand, suppose that $\psi \in H_0^{1/2}(\Delta_R)$. We extend ψ to $\psi_{ext} \in W_0^{1,2}(B_R)$ by solving the problems

$$\text{(D+,D-)}\qquad \begin{cases} \sum_{i=1}^{n+1}\partial_{x_i}^2\psi_{ext}^\pm = 0 \text{ in } B_R^\pm \\ \psi_{ext}^\pm|_{\Delta_R} = \psi,\ \ \psi_{ext}^\pm|_{\partial B_R^\pm \cap \mathbb{R}^{n+1}_\pm} = 0 \end{cases}$$

We set $\psi_{ext} \equiv \psi_{ext}^+ 1_{B_R^+} + \psi_{ext}^- 1_{B_R^-}$, and by standard theory of harmonic functions we have

$$\|\nabla\psi_{ext}\|_{L^2(B_R)} \le C\|\psi\|_{H^{1/2}(\Delta_R)}.$$

Thus, we may define a bounded linear functional on $H_0^{1/2}(\Delta_R)$ by $\Xi_R(\psi) \equiv \Lambda_R(\psi_{ext})$. Since $\Lambda_R(\Upsilon) = 0$ whenever $\Upsilon \in W_0^{1,2}(B_R^+)$, then $\Xi_R(\psi) = \Lambda_R(\Psi)$ for *every* extension $\Psi \in W_0^{1,2}(B_R)$ with $tr(\Psi) = \psi$. Thus, there exists a unique $g_R \in H^{-1/2}(\Delta_R)$ with

$$\iint_{B_R^+} A\nabla u \cdot \overline{\nabla\Psi} = \langle g_R, tr(\Psi)\rangle,\ \ \forall\,\Psi \in W_0^{1,2}(B_R).$$

Now suppose that $R_1 < R_2$, and construct g_{R_k} corresponding to $B_k \equiv B(0, R_k), k = 1, 2$. Then, since $W_0^{1,2}(B_1) \subset W_0^{1,2}(B_2)$ (if we extend elements in the former space to be 0 outside of B_1), we have that $g_{R_1} = g_{R_2}$ in $H^{-1/2}(\Delta_{R_1})$. Thus, $\langle g_{R_1}, \psi\rangle = \langle g_{R_2}, \psi\rangle$, whenever $\psi \in H_c^{1/2}(\mathbb{R}^n)$, and B_1, B_2 contain the support of ψ. It follows that $g \equiv \lim_{R\to 0} g_R$ exists in the sense of distributions, and that

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \overline{\nabla\Psi} = \langle g, tr(\Psi)\rangle,\ \ \forall\,\Psi \in W_c^{1,2}(\mathbb{R}^{n+1}). \tag{4.4}$$

To complete the proof of (*iii*), it remains only to establish that $g \in L^2$. The bound

$$\|g\|_2 \le C\sup_{t>0}\|\nabla u(\cdot,t)\|_2$$

will be an immediate consequence of (*iv*), to which we now turn our attention.

(*iv*). Again we present only the case $\rho = \infty$. Since $\sup_{t>0}\|\nabla u(\cdot,t)\|_2 < \infty$, it is enough to verify the weak convergence for test functions in C_0^∞. Let $\Psi \in C_0^\infty(\mathbb{R}^{n+1})$, $\psi \equiv \Psi|_{\{t=0\}}$. By (4.4), it is enough to show that

$$\int_{\mathbb{R}^n} \vec{N} \cdot A\nabla u(\cdot,t)\psi \to \iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \nabla\Psi,$$

as $t \to 0$. Integrating by parts, we see that for each $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \vec{N} \cdot P_\varepsilon(A\nabla u(\cdot,t))\psi = \iint_{\mathbb{R}^{n+1}_+} P_\varepsilon\left(A\nabla u(\cdot,t+s)\right)(x) \cdot \nabla\Psi(x,s)dxds, \tag{4.5}$$

since $Lu = 0$ and our coefficients are t-independent. By dominated convergence, we may pass to the limit as $\varepsilon \to 0$ in (4.5) to obtain

$$\int_{\mathbb{R}^n} \vec{N} \cdot A\nabla u(\cdot,t)\psi = \iint_{\mathbb{R}^{n+1}_+} A(x)\nabla u(x,t+s) \cdot \nabla\Psi(x,s)dxds, \tag{4.6}$$

It therefore suffices to show that

$$\iint_{\mathbb{R}^{n+1}_+} A(x)(\nabla u(x,t+s) - \nabla u(x,s)) \cdot \nabla\Psi(x,s)dxds = O\left(\sqrt{t}\right), \quad \text{as } t \to 0.$$

To this end, let R denote the radius of a ball centered at the origin which contains the support of Ψ. We split the integral into $\int_0^{2t}\int_{\{|x|<R\}} + \int_{2t}^R\int_{\{|x|<R\}}$. Since $\sup_{t>0}\|\nabla u(\cdot,t)\|_2 < \infty$, the first of these contributes at most $O(t)$, while the second is dominated by

$$C\|\nabla\Psi\|_2\, t\left(\int_t^R \|\nabla\partial_s u(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\right)^{1/2} \leq C_\Phi t\left(\int_{t/2}^\infty \frac{ds}{s^2}\right)^{1/2} \sup_{t>0}\|\nabla u(\cdot,t)\|_2,$$

where in the last step we have used Caccioppoli's inequality in Whitney cubes in the 1/2-space. The desired conclusion follows. □

Next we discuss the boundedness of non-tangential maximal functions of layer potentials. We recall that S_t^η is defined in (2.17), and that P_t denotes a smooth approximate identity acting in $\mathbb{R}^n$. In the sequel, given an operator T, we shall use the notation

$$\|T\|_{op,Q} \equiv \|T\|_{L^2(Q)\to L^2(\mathbb{R}^n)} \equiv \sup \frac{\|Tf\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(Q)}}, \tag{4.7}$$

where the supremum runs over all f supported in Q with $\|f\|_2 > 0$.

Lemma 4.8. *Let L, L^* satisfy the standard assumptions. Then for $1 < p < \infty$, we have*

(i) $\|N_*(\partial_t S_t f)\|_p \leq C_p\left(\sup_{t>0}\|\partial_t S_t\|_{p\to p} + 1\right)\|f\|_p$.
(ii) $\|\widetilde{N}_*(\nabla S_t f)\|_p \leq C_p\left(\sup_{t>0}\|\nabla_x S_t f\|_p + \|N_*(\partial_t S_t f)\|_p\right)$.
(iii) $\|N_*\left(P_t(\nabla S_t f)\right)\|_p \leq C_p\left(\sup_{t>0}\|\nabla_x S_t f\|_p + \|N_*(\partial_t S_t f)\|_p\right)$.
(iv) $\sup_{t_0\geq 0}\|N_*\left(P_t\partial_t S^\eta_{t+t_0} f\right)\|_2 \leq C\left(\sup_{t>0}\|\partial_t S^\eta_t\|_{op,Q} + 1\right)\|f\|_2$, $\eta > 0$, $\operatorname{supp} f \subset Q$.
(v) $\|N_*\left((S_t\nabla)\cdot \mathbf{f}\right)\|_{L^{2,\infty}} \leq C\left(\sup_{t>0}\|(S_t\nabla)\|_{2\to 2} + 1\right)\|\mathbf{f}\|_2$.
(vi) $\|N_*\left(\mathcal{D}_t f\right)\|_{L^{2,\infty}} \leq C\left(\sup_{t>0}\|(S_t\nabla)\|_{2\to 2} + 1\right)\|f\|_2$.

where $L^{2,\infty}$ denotes the usual weak-L^2 space.

Proof. By Lemma 2.2, the kernel $K_t(x,y) \equiv \partial_t\Gamma(x,t,y,0)$ is a standard Calderón-Zygmund kernel with bounds independent of t. We may then prove (i) by a familiar argument involving Cotlar's inequality for maximal singular integrals. We omit the details (but see the proof of (iv) below, which is similar). Estimate (ii) may be obtained by following the argument in [KP], p. 494 (again we omit the details) and (vi) follows from (v). It remains to prove (iii), (iv) and(v).

(iii). The proof is similar to that of estimate (ii), and we follow [KP]. Fix $x_0 \in \mathbb{R}^n$, and suppose that $|x - x_0| < t$. It is enough to replace ∇ by $\nabla_{\|}$. We have

$$\begin{aligned} P_t\left(\nabla_{\|} S_t f\right)(x) &= \nabla_x P_t(S_t f)(x) \equiv t^{-1}\vec{Q}_t(S_t f)(x) \\ &= t^{-1}\vec{Q}_t\left(\int_0^t \partial_s S_s f ds + S_0 f - \fint_{\Delta_{2t}(x_0)} S_0 f\right)(x) \end{aligned}$$

where we have used that $t\nabla_x P_t \equiv \vec{Q}_t$ annihilates constants. But

$$\left|\vec{Q}_t\left(t^{-1}\int_0^t \partial_s S_s f ds\right)(x)\right| \leq CM\left(N_*(\partial_s S_s f)\right)(x_0),$$

and, by Poincaré's inequality,

$$\left|t^{-1}\vec{Q}_t\left(S_0 f - \fint_{\Delta_{2t}(x_0)} S_0 f\right)(x)\right| \leq CM(\nabla_{\|} S_0 f)(x_0).$$

(iv). We suppose that $\eta << \ell(Q)$, and that Q is centered at 0, as it is only this case that we shall encounter in the sequel. We shall deduce (iv) as a consequence of the following refinement of Cotlar's inequality for maximal singular integrals. Let T be a singular integral operator associated to a standard Calderón-Zygmund kernel $K(x,y)$. As usual, we define truncated singular integrals

$$T_\varepsilon f(x) \equiv \int_{|x-y|>\varepsilon} K(x,y) f(y) dy,$$

and we define a maximal singular integral

$$T_*^R f \equiv \sup_{0<\varepsilon<R} |T_\varepsilon f|.$$

We claim that the following holds for all f supported in a cube Q:

$$T_*^{\ell(Q)} f(x) \leq C\left(C_K + \|T\|_{op,Q}\right) Mf(x) + CM(Tf)(x), \tag{4.9}$$

where C_K depends on the Calderón-Zygmund kernel conditions. Momentarily taking this claim for granted, we proceed to prove (iv).

Let $K_t^\eta(x,y)$ denote the kernel of $\partial_t S_t^\eta$ (see (2.17)), i.e.,

$$K_t^\eta(x,y) \equiv \partial_t\left(\varphi_\eta * \Gamma(x,\cdot,y,0)\right)(t).$$

Then by Lemma 2.2 we have for all $t \geq 0$, uniformly in $t_0 \geq 0$,

$$|K_{t+t_0}^\eta(x,y)| \leq C\left(\frac{1_{|x-y|+t>40\eta}}{(t+|x-y|)^n} + \frac{1_{|x-y|+t<40\eta}}{\eta|x-y|^{n-1}}\right) \tag{4.10}$$

$$|K_{t+t_0}^\eta(x+h,y) - K_{t+t_0}^\eta(x,y)| \leq C\frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}}, \quad |x-y|+t > 10\eta \tag{4.11}$$

where the last bound holds whenever $|x-y| > 2|h|$ or $2t > |h|$. Of course, we also have a similar estimate concerning Hölder continuity in the y variable. In particular, $K_{t+t_0}^\eta(x,y)$ is a standard Calderón-Zygmund kernel, uniformly in t, t_0 and η.

We begin by showing that for each fixed $x_0 \in \mathbb{R}^n$ and $t_0 \geq 0$,

$$N_*\left(P_t\partial_t S_{t+t_0}^\eta f\right)(x_0) \leq \sup_{t>0}|\partial_t S_t^\eta f(x_0)| + CM(Mf)(x_0). \tag{4.12}$$

To see this, let $|x - x_0| < t$, and note that

$$|P_t(\partial_t S_{t+t_0}^\eta f)(x) - \partial_t S_{t+t_0}^\eta f(x_0)| \leq Ct^{-n}\int_{|x_0-z|<2t}\int_{\mathbb{R}^n}|K_{t+t_0}^\eta(z,y) - K_{t+t_0}^\eta(x_0,y)||f(y)|dydz,$$

for which, in the case $t > 10\eta$, we obtain immediately the bound $CMf(x_0)$ by applying (4.11). In the case $t \le 10\eta$, we split the inner integral into

$$\int_{|x_0-y|>10\eta} + \int_{|x_0-y|\le 10\eta} \le CMf(x_0) + C\left(Mf(z) + Mf(x_0)\right),$$

where we have applied (4.11) to bound the first term, and (4.10) to handle the second. The estimate (4.12) now follows readily.

Next, we observe that for f supported in a cube Q centered at 0, with $\ell(Q) >> \eta$,

$$\sup_{t>0} |\partial_t S_t^\eta f(x)| \le \sup_{0<t<\ell(Q)} |\partial_t S_t^\eta f(x)| + CMf(x). \tag{4.13}$$

Indeed, suppose that $t \ge \ell(Q) >> \eta$. Then

$$|\partial_t S_t^\eta f(x)| \le \int |K_t^\eta(x,y) f(y)| dy \le CMf(x),$$

by (4.10), since for $y \in Q$, we have $|x-y| \approx |x|$, if $|x| > Ct$, and $|x-y| < Ct$, if $|x| < Ct$.

Combining (4.12) and (4.13), we see that it is enough to treat $\sup_{0<t<\ell(Q)} |\partial_t S_t^\eta f(x)|$. To this end, fix x_0 and $t \in (0,\ell(Q))$, and set $\rho \equiv \max(t, 2\eta)$. Then

$$\begin{aligned}\partial_t S_t^\eta f(x_0) \quad = \quad & \int_{|x_0-y|>5\rho} \left(K_t^\eta(x_0,y) - K_0^\eta(x_0,y)\right) f(y) dy \\ & + \int_{|x_0-y|\le 5\rho} K_t^\eta(x_0,y) f(y) dy - \int_{5\rho>|x_0-y|>\rho} K_0^\eta(x_0,y) f(y) dy \\ & + \int_{|x_0-y|>\rho} K_0^\eta(x_0,y) f(y) dy \equiv I + II + III + IV.\end{aligned}$$

Then $|I| + |II| + |III| \le CMf(x_0)$, by Lemma 2.2 and by (4.10). Also,

$$|IV| \quad \le \quad \sup_{0<\varepsilon<\ell(Q)} \left| \int_{|x_0-y|>\varepsilon} K_0^\eta(x_0,y) f(y) dy \right|.$$

Thus, taking T in (4.9) to be the singular integral operator with kernel $K_0^\eta(x,y)$, we obtain (iv), modulo the proof of (4.9).

We now turn to the proof of (4.9). The argument is a variant of the standard one. Suppose that f is supported in a cube Q, and fix $\varepsilon \in (0,\ell(Q))$ and $x_0 \in \mathbb{R}^n$. Set $\Delta \equiv \Delta_{\varepsilon/2}(x_0)$, $2\Delta \equiv \Delta_\varepsilon(x_0)$. Let $f_1 \equiv f 1_{2\Delta}$, $f_2 \equiv f - f_1$. Then for $x \in \Delta$, we have

$$\begin{aligned}|T_\varepsilon f(x_0)| = |T f_2(x_0)| \quad & = \quad |T f_2(x_0) - T f_2(x) + T f(x) - T f_1(x)| \\ & \le \quad C_K M f(x_0) + |T f(x)| + |T f_1(x)|.\end{aligned}$$

Let $r \in (0,1)$, and take an L^r average of this last inequality over Δ. Note that $f_1 = 0$ unless $2\Delta \subset 5Q$, since $\mathrm{diam}(2\Delta) \le 2\ell(Q)$. We therefore obtain

$$\begin{aligned}|T_\varepsilon f(x_0)| \quad & \le \quad C_K M f(x_0) + M(|Tf|^r)^{1/r}(x_0) + \left(\fint_\Delta |T f_1|^r \right)^{1/r} \\ & \le \quad C\left(C_K + \|T\|_{L^1(Q)\to L^{1,\infty}(5Q)}\right) M f(x_0) + M(Tf)(x_0),\end{aligned}$$

where we have used Kolmogorov's weak-L^1 criterion, and $L^{1,\infty}$ is the usual weak-L^1 space. But by a localized version of the Calderón-Zygmund Theorem,

$$\|T\|_{L^1(Q)\to L^{1,\infty}(5Q)} \le C\left(C_K + \|T\|_{L^2(Q)\to L^2(5Q)}\right) \le C\left(C_K + \|T\|_{L^2(Q)\to L^2(\mathbb{R}^n)}\right),$$

and (4.9) follows.

(v). By (i) and t-independence, we may replace ∇ by ∇_x. The desired estimate is an immediate consequence of the following pointwise bound. For convenience of notation set $\mathbf{K} \equiv \sup_{t>0} \| (S_t \nabla_\|) \|_{2\to 2}$. Let $\vec{f} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$. We shall prove[7]

$$N_*((S_t\nabla_x) \cdot \vec{f})(x) \le C\left(M((S_t|_{t=0}\nabla_x) \cdot \vec{f})(x) + (\mathbf{K}+1)(M(|\vec{f}|^2))^{1/2}(x)\right) \tag{4.14}$$

To this end, we fix $(x_0, t_0) \in \mathbb{R}^{n+1}$ and suppose that $|x_0 - x| < 2t$, $|t_0 - s| < 2t$ and that $k \ge 4$. We claim that

$$\int_{2^k t \le |x_0 - y| < 2^{k+1} t} |\nabla_y(\Gamma(x, s, y, 0) - \Gamma(x_0, t_0, y, 0))|^2 dy \le C 2^{-k\alpha}(2^k t)^{-n}. \tag{4.15}$$

Indeed, the special case $s = t_0$ is essentially a reformulation of Lemma 2.13, but with the roles of x and y reversed. In general, we write

$$\Gamma(x, s, y, 0) - \Gamma(x_0, t_0, y, 0) = \{\Gamma(x, s, y, 0) - \Gamma(x_0, s, y, 0)\} + \{\Gamma(x_0, s, y, 0) - \Gamma(x_0, t_0, y, 0)\}.$$

The first expression in brackets is the case $s = t_0$, while the horizontal gradient of the second equals

$$\int_{t_0}^{s} \nabla_y \partial_\tau \Gamma(x_0, \tau, y, 0) d\tau.$$

We may handle the contribution of the latter term via Lemma 2.5. This proves the claim.

We set $u(\cdot, t) \equiv (S_t\nabla_\|) \cdot \vec{f}$, and we split $u = u_0 + \sum_{k=4}^\infty u_k \equiv u_0 + \tilde{u}$, where

$$u_0 \equiv (S_t\nabla_\|) \cdot \vec{f}_0, \quad u_k \equiv (S_t\nabla_\|) \cdot \vec{f}_k, \quad \tilde{u} \equiv \sum_{k=4}^\infty u_k,$$

and $\vec{f}_0 \equiv \vec{f} 1_{\{|x_0 - \cdot| < 16t\}}$, $\vec{f}_k = \vec{f} 1_{\mathcal{R}_k}$, and $\mathcal{R}_k \equiv \{y : 2^k t \le |x_0 - y| < 2^{k+1} t\}$. By (4.15), for $s \in [-2t, 2t]$ and $|x_0 - x| < 2t$, we have that

$$|u_k(x, s) - u_k(x_0, 0)| \le C 2^{-k\alpha/2} \left(\fint_{\mathcal{R}_k} |\vec{f}|^2 \right)^{1/2} \le C 2^{-k\alpha/2} \left(M(|\vec{f}|^2) \right)^{1/2} (x_0).$$

Summing in k, we obtain

$$|\tilde{u}(x, s) - \tilde{u}(x_0, 0)| \le C \left(M(|\vec{f}|^2) \right)^{1/2} (x_0). \tag{4.16}$$

Moreover, since $Lu_0 = 0$, by (1.3) it follows that

$$\begin{aligned} |u_0(x, t)| \;&\le\; C\left(\fint\!\!\fint_{B((x,t),t/2)} |u_0|^2 \right)^{1/2} \;\le\; C t^{-n/2} \sup_{\tau>0} \| (S_\tau \nabla_\|) \cdot \vec{f}_0 \|_2 \\ &\le\; C\mathbf{K} \left(M(|\vec{f}|^2) \right)^{1/2} (x_0). \end{aligned}$$

Taking $s = t$ in (4.16), we therefore need only establish the bound

$$|\tilde{u}(x_0, 0)| \le C(\mathbf{K}+1) \left(M(|\vec{f}|^2) \right)^{1/2} (x_0) + CM\left(u(\cdot, 0)\right)(x_0) \tag{4.17}$$

The proof of (4.17) is based on that of the well known Cotlar inequality for maximal singular integrals. Set $\Delta_0 = \{|x - x_0| < t\}$, and let $x \in \Delta_0$. We write

$$\begin{aligned} |\tilde{u}(x_0, 0)| &\le |\tilde{u}(x, 0) - \tilde{u}(x_0, 0)| + |\tilde{u}(x, 0)| \\ &\le |\tilde{u}(x, 0) - \tilde{u}(x_0, 0)| + |u_0(x, 0)| + |u(x, 0)| \\ &\le C \left(M(|\vec{f}|^2) \right)^{1/2} (x_0) + |u_0(x, 0)| + |u(x, 0)|, \end{aligned}$$

[7]The bound for the last term in (4.14) may be improved to $(M(|\vec{f}|^q))^{1/q}(x)$, for some $q < 2$ depending on dimension and ellipticity, as the fourth named author will show in a forthcoming paper with M. Mitrea.

where in the last step we have used (4.16) with $s = 0$. Averaging over Δ_0, we obtain

$$|\tilde{u}(x_0, 0)| \leq C\left(M(|\vec{f}|^2)\right)^{1/2}(x_0) + \left(\fint_{\Delta_0} |u_0(x, 0)|^2 dx\right)^{1/2} + M\left(u(\cdot, 0)\right)(x_0).$$

Since the L^2 average of u_0 is bounded by $C\mathbf{K}\left(M(|\vec{f}|^2)\right)^{1/2}(x_0)$, we obtain (4.17). □

We are now ready to discuss the jump relations and traces of the layer potentials. We recall that $S_t^*, \mathcal{D}_t^*$ denote the single and double layer potentials associated to L^*.

Lemma 4.18. *Suppose that L, L^* satisfy the standard assumptions, and that the single layer potentials S_t, S_t^* satisfy*

$$\sup_{t\neq 0} \|\nabla S_t\|_{2\to 2} + \sup_{t\neq 0} \|\nabla S_t^*\|_{2\to 2} < \infty. \tag{4.19}$$

Then there exist L^2 bounded operators $K, \widetilde{K}, \mathcal{T}$ with the following properties: for all $f \in L^2(\mathbb{R}^n)$, we have

(i) $\left(\pm\frac{1}{2}I + \widetilde{K}\right) f = \partial_\nu u^\pm$

where $u^\pm \equiv S_t f$, $t \in \mathbb{R}_\pm$, and ∂_ν denotes the conormal derivative $-e_{n+1} \cdot A\nabla$, interpreted in the weak sense of Lemma 4.3 (iii) and (iv).

(ii) $\mathcal{D}_{\pm s} f \to \left(\mp\frac{1}{2}I + K\right) f$ *weakly in L^2*

(iii) $(\nabla S_t)|_{t=\pm s} f \to \left(\mp\frac{1}{2A_{n+1,n+1}} e_{n+1} + \mathcal{T}\right) f$ *weakly in L^2.*

Proof. It is enough to prove (i). Indeed, if we define

$$K \equiv adj\left(\widetilde{K^*}\right),$$

then (ii) follows from (i) and the observation that $\mathcal{D}_s = adj\left(\vec{N} \cdot A^*\nabla S_t^*\right)|_{t=-s}$. To obtain (iii), we first use (4.19), Lemma 4.8, Lemma 4.3 and the formula

$$-A_{n+1,n+1}\partial_t S_t = \vec{N} \cdot A\nabla S_t + \sum_{j=1}^{n} A_{n+1,j} D_j S_t \tag{4.20}$$

to deduce that $\partial_t S_t f$ converges weakly in L^2, as $t \to 0$. Thus, we may define

$$\mathcal{T} f \equiv tr\left(\nabla S_t f\right).$$

Then (iii) follows from (4.20) since $\nabla_{\|} S_t f$ does not jump across the boundary.

To prove (i), we apply Lemma 4.3 (iii) in both $\mathbb{R}^{n+1}_\pm$, to obtain $g^\pm \in L^2(\mathbb{R}^n)$, with $g^\pm = \partial_\nu u^\pm$ in the weak sense. We now define[8] $\widetilde{K}$ by

$$\left(\frac{1}{2}I + \widetilde{K}\right) f \equiv g^+, \;\; \left(-\frac{1}{2}I + \widetilde{K}\right) f \equiv g^-, \tag{4.21}$$

and to show that this operator is well defined, we need only verify that $g^+ - g^- = f$. It is enough to prove that

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla u^+ \cdot \nabla\Psi dxdt + \iint_{\mathbb{R}^{n+1}_-} A\nabla u^- \cdot \nabla\Psi dxdt = \int_{\mathbb{R}^n} f\Psi dx, \tag{4.22}$$

for all $\Psi \in C_0^\infty(\mathbb{R}^{n+1})$. To this end, set $u^\pm_\eta \equiv S_t^\eta f$, where S_t^η is defined in (2.17), so that

$$u^\pm_\eta = \iint_{\mathbb{R}^{n+1}} \Gamma(x, t, y, s) f_\eta(y, s) dyds, \quad t \in \mathbb{R}_\pm$$

[8]We are indebted to M. Mitrea for suggesting this approach.

where $f_\eta(y,s) \equiv f(y)\varphi_\eta(s)$ and φ_η is the kernel of a smooth approximate identity acting in 1 dimension. Let $U_\eta \equiv u_\eta^+ 1_{\mathbb{R}^{n+1}_+} + u_\eta^- 1_{\mathbb{R}^{n+1}_-}$. Since $L\Gamma = \delta$, we have that

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}_+} A\nabla u_\eta^+ \cdot \nabla\Psi + \iint_{\mathbb{R}^{n+1}_-} A\nabla u_\eta^- \cdot \nabla\Psi &= \iint_{\mathbb{R}^{n+1}} A\nabla U_\eta \cdot \nabla\Psi \\ &= \iint_{\mathbb{R}^{n+1}} f_\eta \Psi \to \int_{\mathbb{R}^n} f\Psi,\end{aligned}$$

as $\eta \to 0$. On the other hand, fixing ε momentarily, we have that

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla(u_\eta^+ - u^+)\cdot\nabla\Psi = \int_\varepsilon^\infty \int_{\mathbb{R}^n} + \int_0^\varepsilon \int_{\mathbb{R}^n} \equiv I_\varepsilon + II_\varepsilon.$$

Fix a number R greater than the diameter of supp(Ψ). Then

$$|I_\varepsilon| \le C_\Psi \int_\varepsilon^R \sup_{\varepsilon<t<R} \|\nabla(S_t^\eta - S_t) f\|_{L^2(\mathbb{R}^n)} \to 0$$

as $\eta \to 0$, by Lemma 2.18. Moreover,

$$\sup_{\eta>0} |II_\varepsilon| \le C_\Psi\, \varepsilon \sup_{t\neq 0} \|\nabla S_t f\|_2 \le C_\Psi\, \varepsilon \|f\|_2,$$

where we have used that $\sup_{\eta>0}\|\nabla S_t^\eta f\|_2 \le \sup_t \|\nabla S_t f\|_2$, by construction of S_t^η (2.17). The analogous convergence result for the lower half-space concludes the proof of (i). □

We turn now to the issues of non-tangential and strong L^2 convergence for $\mathcal{D}_t$.

Lemma 4.23. *Suppose that L, L^* satisfy the standard assumptions, that the single layer potentials S_t, S_t^* satisfy* (4.19)*, and that $S_0^* \equiv S_t^*|_{t=0} : L^2(\mathbb{R}^n) \to \dot{L}^2_1(\mathbb{R}^n)$ is bijective. Then for every $f \in L^2(\mathbb{R}^n)$, we have the following:*

$$\mathcal{D}_{\pm t} f \to \left(\mp\frac{1}{2}I + K\right) f \quad \textit{n.t. and in } L^2.$$

We first require a special case of the Gauss-Green formula.

Lemma 4.24. *Let L, L^* satisfy the standard assumptions, and suppose that $Lu = 0$, $L^*w = 0$ in $\mathbb{R}^{n+1}_+$ with*

$$\sup_{t>0} (\|\nabla u(\cdot,t)\|_2 + \|\nabla w(\cdot,t)\|_2) < \infty, \tag{4.25}$$

and $\partial_\nu u\, \overline{w}(\cdot,0)$, $\overline{\partial_{\nu^*} w}\, u(\cdot,0) \in L^1(\mathbb{R}^n)$[9]*. Suppose also that there exist $R_0, \beta > 0$ such that for all $R > R_0$, we have*

$$\iint_{\mathbb{R}^{n+1}_+ \cap (B(0,2R)\setminus B(0,R))} |\nabla u||\nabla w| + |\nabla u| R^{-1}|w| + |\nabla w| R^{-1}|u| = O\left(R^{-\beta}\right). \tag{4.26}$$

Then

$$\int_{\mathbb{R}^n} \partial_\nu u\, \overline{w} = \int_{\mathbb{R}^n} u\, \overline{\partial_{\nu^*} w}.$$

Of course, the analogous result holds in $\mathbb{R}^{n+1}_-$.

[9]Here, ∂_ν and ∂_{ν^*} are the exterior conormal derivatives, corresponding to the matrices A and A^* respectively, which exist in the weak sense of Lemma 4.3.

Proof. By the symmetry of our hypotheses, it is enough to show that

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \overline{\nabla w} = \int_{\mathbb{R}^n} \partial_\nu u\, \overline{w}. \tag{4.27}$$

To this end, for $R_0 < R < \infty$, let $\Theta_R(X) \equiv \Theta(X/R)$, where $\Theta \in C_0^\infty(B(0,2))$ and $\Theta \equiv 1$ in $B(0,1)$. We set $w_R \equiv w\Theta_R$. Then by Lemma 4.3, we have that

$$\iint_{\mathbb{R}^{n+1}_+} A\nabla u \cdot \overline{\nabla w_R} = \int_{\mathbb{R}^n} \partial_\nu u\, \overline{w_R}.$$

A simple limiting argument completes the proof. □

Corollary 4.28. *Let L, L^* satisfy the standard assumptions, and suppose that the respective single layer potentials S_t, S_t^* satisfy* (4.19). *Further suppose that $u(\cdot,\tau) = S_\tau\psi$ in $\mathbb{R}^{n+1}_-$, where $\psi \in C_0^\infty(\mathbb{R}^n)$. Then setting $u_0 \equiv u(\cdot,0)$, we have*

$$\mathcal{D}_t u_0 = S_t(\partial_\nu u). \tag{4.29}$$

Proof. It is enough to show that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \mathcal{D}_t u_0\, \overline{\varphi} = \int_{\mathbb{R}^n} S_t(\partial_\nu u)\, \overline{\varphi}.$$

Note that $adj(\mathcal{D}_t) = \vec{N} \cdot A^* \, (\nabla S_\tau^*)\,|_{\tau=-t}$, and that $adj(S_t) = S_{-t}^*$. Set $u^*(\cdot,\tau) \equiv S_\tau^*\varphi$,, so that $L^*u^* = 0$ in $\mathbb{R}^{n+1} \setminus \{\tau = 0\}$. It suffices to verify the hypotheses of Lemma 4.24, in the lower half-space, for u, w, with $w(\cdot,s) \equiv u^*(\cdot, s-t),\ s \leq 0$. Estimate (4.25) is immediate by (4.19). By Lemma 2.2, we have

$$|u(X)| + |w(X)| = O(|X|^{-n+1}) \quad \text{as } |X| \to \infty. \tag{4.30}$$

Also, $Lu = 0, L^*w = 0$ in $\mathbb{R}^{n+1} \setminus B(0,R_0)$, if R_0 is chosen large enough, since φ, ψ have compact support. Thus, by Caccioppoli,

$$\iint_{\mathbb{R}^{n+1}_- \cap (B(0,2R)\setminus B(0,R))} |\nabla u|^2 \leq C \iint_{\mathbb{R}^{n+1}_- \cap (B(0,3R)\setminus B(0,R/2))} \left(\frac{|u|}{R}\right)^2 = O\left(R^{-n+1}\right),$$

for $R > 4R_0$, and similarly for w. Estimate (4.26) follows. Finally, the boundary integrability of $\partial_\nu u\, \overline{w}$ and $\overline{\partial_{\nu^*} w}\, u$ follows readily from Cauchy-Schwarz, the fact that $n \geq 2$, and two observations: first, that by Lemma 2.7 and duality, we have

$$\int_{\Delta_{2R}(0)\setminus\Delta_R(0)} |\partial_\nu u|^2 + |\partial_{\nu^*} w|^2 = O(R^{-n});$$

second, that (4.30) implies that

$$\int_{\Delta_{2R}(0)\setminus\Delta_R(0)} |u|^2 + |w|^2 = O\left(R^{2-n}\right).$$

We leave the remaining details to the reader. □

Proof of Lemma 4.23. Since we have already obtained the limits $(\mp\frac{1}{2}I + K)f$ in the weak sense (Lemma 4.18), it is enough here merely to establish existence of *n.t.* and strong L^2 limits, without concern for their precise values. We give the proof only in the case of the upper half-space, as the proof in the other case is the same.

We begin with the matter of non-tangential convergence. Observe that $adj\,(S_t\nabla) = (\nabla S_\tau^*)\,|_{\tau=-t}$, so by (4.19) and Lemma 4.8(*vi*), it is enough to establish *n.t.* convergence for f in a dense class in L^2. We claim now that $\{S_0 \operatorname{div}_\| \vec{g} : \vec{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)\}$ is dense in L^2.

Indeed, by hypothesis and duality, $S_0 : \dot{L}^2_{-1} \to L^2$ is bijective. Thus, $L^2 = \{S_0 \operatorname{div}_{\|} \vec{g} : \vec{g} \in L^2\}$. The density of C_0^∞ in L^2 establishes the claim.

We now set $f = u_0 = S_0(\operatorname{div}_{\|} \vec{g})$, with $\vec{g} \in C_0^\infty$, and let $u(\cdot, \tau) = S_\tau(\operatorname{div}_{\|} \vec{g})$, $\tau < 0$. We may then apply Corollary 4.28 to obtain that $\mathcal{D}_t f = S_t(\partial_\nu u)$. Moreover, (4.19), Lemma 4.8, and Lemma 4.3 imply that $\partial_\nu u \in L^2$ and hence also that $S_t(\partial_\nu u)$ converges *n.t.*, from which fact the non-tangential part of (*ii*) now follows.

We turn now to the issue of strong convergence in L^2. By (4.19), we have in particular that L^2 bounds hold, uniformly in $t > 0$, for $\mathcal{D}_t$. Thus, it is once again enough to establish convergence in a dense class. To this end, choose u_0, u as above. It suffices to show that $\mathcal{D}_t u_0$ is Cauchy convergent in L^2, as $t \to 0$. Suppose that $0 < t' < t \to 0$, and observe that, by Corollary 4.28, (4.19) and our previous observation that $\partial_\nu u \in L^2$,

$$\|\mathcal{D}_t u_0 - \mathcal{D}_{t'} u_0\|_2 = \| \int_{t'}^{t} \partial_s S_s(\partial_\nu u)\, ds\|_2 \leq (t - t')\|\partial_s S_s \partial_\nu u\|_2 \to 0.$$

$\square$

Lemma 4.31. *(**Uniqueness**). Suppose that L, L^* satisfy the standard assumptions, and that we have existence of solutions to (D2) and (R2). Then those solutions are unique, in the following sense:*

(i) *If u solves (D2), with $u(\cdot, t) \to 0$ in L^2, as $t \to 0$, then $u \equiv 0$.*
(ii) *If u solves (R2), and $u \to 0$ n.t., then $u \equiv 0$.*[10]

If, in addition, L and L^* have "Good Layer Potentials", then the solution to (N2) is unique, in the sense that:

(iii) *If u solves (N2), with $\partial u/\partial\nu = 0$ in the sense of Lemma 4.3 (iii) and (iv), then $u \equiv 0$ (modulo constants).*

Proof. Consider first uniqueness in (D2). We begin by constructing Green's function. By Lemma 2.5 with $m = -1$, for each fixed $(x, t) \in \mathbb{R}^{n+1}$, we have $\Gamma(x, t, \cdot, 0) \in \dot{L}^2_1$, with

$$\|\nabla_{\|}\Gamma(x, t, \cdot, 0)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-n/2}. \tag{4.32}$$

Thus, by (R2), there exists $w = w_{x,t}$ solving

$$\begin{cases} Lw = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ w(\cdot, s) \to \Gamma(x, t, \cdot, 0)\, n.t. \\ \|\widetilde{N}_*(\nabla w)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-n/2}. \end{cases} \tag{R2}$$

Set

$$G(x, t, y, s) \equiv \Gamma(x, t, y, s) - w_{x,t}(y, s),$$

and note that

$$\sup_{s:|s-t|>t/8} \|\nabla G(x, t, \cdot, s)\|_{L^2(\mathbb{R}^n)} \leq Ct^{-n/2}. \tag{4.33}$$

Let $\theta \in C_0^\infty(\mathbb{R}^{n+1}_+)$, with $\theta \equiv 1$ in a neighborhood of (x, t). Then, since $Lu = 0$, we have

$$\begin{aligned} u(x, t) = (u\theta)(x, t) &= \iint \overline{A^* \nabla_{y,s} G(x, t, y, s)} \cdot \nabla(u\theta) dyds \\ &= -\iint \overline{G}\, \nabla\theta \cdot A\nabla u + \iint \overline{\nabla G} \cdot A\nabla\theta\, u \equiv I + II. \end{aligned}$$

[10] Our data in the problem (R2) belongs to $\dot{L}^2_1$, whose elements are defined modulo constants; thus, uniqueness in this context must be interpreted correspondingly. We assume here that we have chosen a particular realization of the data equal to 0 *a.e.* on the boundary.

We now choose $\phi \in C_0^\infty(-2,2)$, $\phi \equiv 1$ in $(-1,1)$, with $0 \le \phi \le 1$, and set $\theta(y,s) \equiv [1-\phi(s/\varepsilon)]\,\phi(s/(100R))\,\phi(|x-y|/R)$, with $\varepsilon < t/8, R > 8t$. With this choice of θ, the domains of integration in I and II are contained in a union $\Omega_1 \cup \Omega_2 \cup \Omega_3$, where

(1) $\Omega_1 \subset \Delta_{2R}(x) \times \{\varepsilon < s < 2\varepsilon\}$, with $\|\nabla\theta\|_{L^\infty(\Omega_1)} \le C\varepsilon^{-1}$.
(2) $\Omega_2 \subset \Delta_{2R}(x) \times \{100R < s < 200R\}$, with $\|\nabla\theta\|_{L^\infty(\Omega_2)} \le CR^{-1}$.
(3) $\Omega_3 \subset (\Delta_{2R}(x) \setminus \Delta_R(x)) \times \{0 < s < 200R\}$. with $\|\nabla\theta\|_{L^\infty(\Omega_3)} \le CR^{-1}$.

We treat term I first. We recall from [HK2] that

$$\|\nabla_{(\cdot)} G(X,\cdot)\|_{L^2(\mathbb{R}^{n+1}_+ \setminus B(X,r))} \le C r^{(1-n)/2}, \quad \forall r > 0,\ X \in \mathbb{R}^{n+1}_+ \tag{4.34}$$

and that

$$|G(X,Y)| \le C|X-Y|^{1-n}, \tag{4.35}$$

whenever $|X-Y| \le \min(\delta(X),\delta(Y))$, where $\delta(X)$ denotes the distance to the boundary of the half-space (i.e., the t-coordinate). Thus, in particular we obtain that

$$R^{-1}\|G(x,t,\cdot)\|_{L^2(\Omega_2\cup\Omega_3)} \le CR^{(1-n)/2}, \tag{4.36}$$

where in proving the bound on Ω_3 we have used that G vanishes on the boundary, to reduce matters to (4.34). We then have that

$$\frac{1}{C}|I| \le \varepsilon^{-1}\iint_{\Omega_1}|G||\nabla u| + R^{(1-n)/2}\left(\iint_{\Omega_2\cup\Omega_3}|\nabla u|^2\right)^{1/2} \equiv I_1 + I_2. \tag{4.37}$$

Since u vanishes on $\{t=0\}$, we may apply Caccioppoli's inequality in $\Omega_2 \cup \Omega_3$ to obtain that $I_2 \le CR^{-n/2}\sup_{s>0}\|u(\cdot,s)\|_2 \to 0$ as $R \to \infty$.

To treat I_1, we first note that for $(y,s) \in \Omega_1$,

$$|G(x,t,y,s)| \le C\varepsilon\left((|x-y|+t)^{-n} + \widetilde{N}_*(\nabla w_{x,t})(y)\right), \tag{4.38}$$

by Lemma 2.2, Lemma 4.3 and construction of G. Consequently,

$$\left(\varepsilon^{-1}\iint_{L^2(\Omega_1)}|G(x,t,y,s)|^2 dyds\right)^{1/2} \le C\varepsilon\, t^{-n/2}, \tag{4.39}$$

Thus, using Caccioppoli to estimate the L^2 norm of ∇u in Ω_1, we obtain that

$$I_1 \le Ct^{-n/2}\sup_{s<3\varepsilon}\|u(\cdot,s)\|_2 \to 0$$

as $\varepsilon \to 0$, since $u(\cdot,s) \to 0$ in L^2.

We now consider term II. By Cauchy-Schwarz and then Caccioppoli's inequality,

$$\begin{aligned} |II| &\le \varepsilon^{-1}\iint_{\Omega_1}|\nabla G|\,|u| + R^{-1}\iint_{\Omega_2\cap\Omega_3}|\nabla G|\,|u| \equiv II_1 + II_2 \\ &\le C\varepsilon^{-3/2}\|G(x,t,\cdot,\cdot)\|_{L^2(\Omega_1)}\sup_{s<2\varepsilon}\|u(\cdot,s)\|_2 \\ &\qquad + R^{-3/2}\|G(x,t,\cdot,\cdot)\|_{L^2(\Omega_2\cup\Omega_3)}\sup_{s>0}\|u(\cdot,s)\|_2. \end{aligned} \tag{4.40}$$

By (4.39), the term II_1 may be handled exactly like I_1, and by (4.36), II_2 yields the same bound as I_2. The proof of uniqueness in (D2) is now complete.

Uniqueness in (R2). Suppose now that $\widetilde{N}_*(\nabla u) \in L^2$, and that $u \to 0$ *n.t.*. Choosing θ as above, we split $u(x,t) = (u\theta)(x,t)$ into the same terms $I + II$, which we dominate again by $I_1 + I_2$ and $II_1 + II_2$ as in (4.37) and (4.40), respectively. We now claim that

$$I_1 + II_1 \le C\varepsilon t^{-n/2}\|\widetilde{N}_*(\nabla u)\|_2 \to 0$$

as $\varepsilon \to 0$. For I_1, this follows from Cauchy-Schwarz and (4.39). To handle II_1, we first note that, by Lemma 4.3(i), $|u(y,s)| \leq C\varepsilon \widetilde{N}_*(\nabla u)(y)$ in Ω_1, since $u(\cdot,0) = 0$ *a.e.*. The claim then follows from Cauchy-Schwarz and Caccioppoli (applied to ∇G).

Rewriting the last expression in (4.37), we see that

$$I_2 = R^{(2-n)/2}\left(R^{-1}\iint_{\Omega_2\cup\Omega_3}|\nabla u|^2\right)^{1/2} \leq CR^{(2-n)/2}\|\widetilde{N}_*(\nabla u)\|_{L^2(\Delta_{2R}(x)\setminus\Delta_R(x))},$$

by construction of $\Omega_2\cup\Omega_3$. Moreover, Lemma 4.3(i) implies that $|u|/R \leq C\widetilde{N}_*(\nabla u)$ in Ω_3 and $|u|/R \leq C\inf_{\Delta_{2R}(x)\setminus\Delta_R(x)}\widetilde{N}_*(\nabla u)$ in Ω_2. Thus, by (4.34),

$$\begin{aligned} II_2 &\leq CR^{1/2}\left(\iint_{\Omega_2\cup\Omega_3}|\nabla_{y,s}G(x,t,y,s)|^2 dyds\right)^{1/2}\left(R^{-1}\iint_{\Omega_2\cup\Omega_3}\frac{|u|}{R}\right)^{1/2} \\ &\leq CR^{(2-n)/2}\,\|\widetilde{N}_*(\nabla u)\|_{L^2(\Delta_{2R}(x)\setminus\Delta_R(x))}. \end{aligned}$$

Since $n \geq 2$, we obtain dominated convergence to 0.

Uniqueness in (N2). Suppose that $\widetilde{N}_*(\nabla u) \in L^2$, and that $\partial u/\partial_\nu = 0$, where the latter is interpreted in the sense of Lemma 4.3(iii) and (iv). By Lemma 4.3(i), we have that $u \to u_0$ *n.t.*, for some $u_0 \in \dot{L}^2_1(\mathbb{R}^n)$. By uniqueness in (R2),

$$u(\cdot,t) = S_t(S_0^{-1}u_0),$$

where $S_0 \equiv S_t|_{t=0}$. Thus, by Lemma 4.18,

$$0 = \frac{\partial u}{\partial \nu} = \left(\frac{1}{2}I + \widetilde{K}\right)(S_0^{-1}u_0).$$

But by hypothesis, $\frac{1}{2}I + \widetilde{K} : L^2 \to L^2$ and $S_0 : L^2 \to \dot{L}^2_1$ are bijective, so that $u_0 = 0$ in the sense of $\dot{L}^2_1$, i.e., $u_0 \equiv$ *constant a.e.*. By uniqueness in (R2), $u \equiv$ *constant*.

□

As a corollary of uniqueness, we shall obtain the following "Fatou Theorem".

Corollary 4.41. *Let L, L^* satisfy the standard assumptions, and have "Good Layer Potentials". Suppose also that $Lu = 0$, and that*

$$\sup_{t>0}\|u(\cdot,t)\|_2 < \infty. \tag{4.42}$$

Then $u(\cdot,t)$ converges n.t. and in L^2 as $t \to 0$.

Proof. By Lemma 4.23, it is enough to show that $u(\cdot,t) = \mathcal{D}_t h$ for some $h \in L^2(\mathbb{R}^n)$. We follow the argument in [St2], pp. 199-200, substituting $\mathcal{D}_t$ for the classical Poisson kernel. For each $\varepsilon > 0$, set $f_\varepsilon \equiv u(\cdot,\varepsilon)$. Let u_ε be the layer potential solution with data f_ε; i.e.,

$$u_\varepsilon(x,t) \equiv \mathcal{D}_t\left[\left(-\frac{1}{2}I + K\right)^{-1} f_\varepsilon\right](x).$$

We claim that $u_\varepsilon(x,t) = u(x,t+\varepsilon)$.

Proof of Claim. Set $U_\varepsilon \equiv u(x,t+\varepsilon) - u_\varepsilon(x,t)$. We observe that

(1) $LU_\varepsilon = 0$ in $\mathbb{R}^{n+1}_+$ (by t-independence of coefficients).
(2) (4.42) holds for U_ε, uniformly in $\varepsilon > 0$
(3) $U_\varepsilon(\cdot,0) = 0$ and $U_\varepsilon(\cdot,t) \to 0$ *n.t.* and in L^2.

(Item (3) relies on interior continuity (1.2) and smoothness in t, along with Lemma 4.23). The claim now follows by Lemma 4.31. □

We return now to the proof of the Corollary. By (4.42), $\sup_\varepsilon \|f_\varepsilon\|_2 < \infty$. Hence, there exists a subsequence f_{ε_k} converging in the weak* topology to some $f \in L^2$. For arbitrary $g \in L^2$, set $g_1 \equiv adj\left(-\frac{1}{2}I + K\right)^{-1} adj(\mathcal{D}_t)g$, and observe that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{D}_t\left(-\frac{1}{2}I+K\right)^{-1} f\,\overline{g} &= \int_{\mathbb{R}^n} f\,\overline{g_1} = \lim_{k\to\infty}\int_{\mathbb{R}^n} f_{\varepsilon_k}\overline{g_1}\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} \mathcal{D}_t\left(-\frac{1}{2}I+K\right)^{-1} f_{\varepsilon_k}\,\overline{g}\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^n} u(\cdot, t+\varepsilon_k)\,\overline{g} = \int_{\mathbb{R}^n} u(\cdot,t)\,\overline{g}.\end{aligned}$$

Since g was arbitrary, the desired conclusion follows. □

We conclude this section with a discussion of $n.t.$ convergence of gradients.

Lemma 4.43. *Suppose that L, L^* satisfy the standard assumptions, and have "Good Layer Potentials". Then for all $f \in L^2$, we have*

$$P_s\left((\nabla S_t)\,|_{t=\pm s}\right) f \to \left(\mp\frac{1}{2A_{n+1,n+1}}e_{n+1} + \mathcal{T}\right) f \quad n.t. \text{ and in } L^2.$$

Proof. We treat only the case of the upper half space, as the proof in the other case is the same. Since the weak limit has already been established (Lemma 4.18) for ∇S_t, it is a routine matter to verify that the strong and $n.t.$ limits for $P_t(\nabla S_t)$ will take the same value, once the existence of those limits has been established. It is to this last point that we therefore turn our attention. By Lemma 4.8 and the dominated convergence theorem, it is enough to establish $n.t.$ convergence.

The non-tangential convergence of $\partial_t S_t$ follows immediately from the "Fatou Theorem" just proved; a simple real variable argument yields the same conclusion for $P_t\partial_t S_t$. We may therefore replace ∇ by $\nabla_\|$. On the other hand, we shall still need to consider the boundary trace of $\partial_t S_t f$, which for the duration of this proof we denote by Vf. Fix now $x_0 \in \mathbb{R}^n$. For $|x - x_0| < t$, we write

$$\begin{aligned}P_t\left(\nabla_\| S_t f\right)(x) &= \nabla_x P_t\left(\int_0^t \partial_s S_s f ds\right)(x) + P_t\left(\nabla_\| S_0 f\right)(x)\\ &\equiv \vec{Q}_t\left(\frac{1}{t}\int_0^t \partial_s S_s f ds\right)(x) + P_t\left(\nabla_\| S_0 f\right)(x) \equiv I + II,\end{aligned}$$

where $\vec{Q}_t 1 = 0$. By standard facts for approximate identities, $II \to \nabla_\| S_0 f$ $n.t.$. Also,

$$I = \vec{Q}_t\left(\frac{1}{t}\int_0^t \left(\partial_s S_s f - Vf\right) ds\right)(x) + \vec{Q}_t\left(Vf - Vf(x_0)\right)(x) \equiv I_1 + I_2.$$

It is straightforward to verify that $I_2 \to 0$ as $t \to 0$, if x_0 is a Lebesgue point for the L^2 function Vf. The term I_1 is more problematic. We first observe that by Lemma 4.3,

$$\left|\vec{Q}_t\left(\frac{1}{t}\int_0^t \left(S_s f - S_0 f\right) ds\right)(x)\right| \leq CtM\left(\widetilde{N}_*(\nabla S_t f)\right)(x_0) \to 0 \tag{4.44}$$

for $a.e.$ x_0. Thus also for $\vec{f} \in C_0^\infty(\mathbb{R}^n)$, we have

$$\left|\vec{Q}_t\left(\frac{1}{t}\int_0^t \left((S_s\nabla_\|)\cdot\vec{f} - (S_0\nabla_\|)\cdot\vec{f}\right) ds\right)(x)\right| \to 0 \ n.t.. \tag{4.45}$$

By Lemma 4.8(v), the density of C_0^∞ in L^2, and the fact that $\vec{Q}_t$ is dominated by the Hardy-Littlewood maximal operator which is bounded from $L^{2,\infty}$ to itself, the latter convergence continues to hold for $\vec{f} \in L^2$. Moreover, if u_0 belongs to the dense class $\{S_0 \operatorname{div}_\| \vec{g} : \vec{g} \in C_0^\infty\}$, by Corollary 4.28 and (4.44), we have that

$$\left|\vec{Q}_t\left(\frac{1}{t}\int_0^t (\mathcal{D}_t u_0 - tr(\mathcal{D}_t u_0))\, ds\right)(x)\right| \to 0 \ n.t., \tag{4.46}$$

and again this fact remains true for u_0 in L^2, by Lemma 4.8(vi) and our previous observation concerning the action of the maximal operator on weak L^2. Combining (4.45) and (4.46) with the adjoint version of the identity (4.20), we obtain convergence to 0 for the term I_1 since every $f \in L^2$ can be written in the form $f = A^*_{n+1,n+1}h$, $h \in L^2$. □

5. P T 1.11:

As noted above, the De Giorgi-Nash estimate (1.2) is stable under L^∞ perturbation of the coefficients. Thus, for ϵ_0 sufficiently small, solutions of $L_1 u = 0$, $L_1^* w = 0$ satisfy (1.2) and (1.3). In particular, the results of Section 2 apply to the fundamental solutions and layer potentials Γ_0, S_t^0 and Γ_1, S_t^1 corresponding to L_0 and L_1, respectively.

We claim that the conclusion of Theorem 1.11 will follow, once we have proved

$$|||t\nabla\partial_t S_t^1|||_{op} \ + \ \sup_{t>0} \|\nabla S_t^1\|_{2\to 2} \leq C \tag{5.1}$$

(recall that $\nabla \equiv \nabla_{x,t}$). Indeed, by the symmetry of our hypotheses, similar bounds will then hold in the lower half space, and for $S_t^{L_1^*}$. Now, by t-independence, $-(S_t^1 D_{n+1}) = D_{n+1}S_t^1$. Moreover, if $\mathcal{J}_t(x,y)$ denotes the kernel of $(S_t^1\nabla_\|)$, and Γ_1^* is the fundamental solution for the adjoint operator L_1^*, then the kernel of $adj(S_t^1\nabla_\|)$ is

$$\overline{\mathcal{J}_t(y,x)} = \nabla_x\overline{\Gamma_1(y,t,x,0)} = \nabla_x\Gamma_1^*(x,0,y,t) = \nabla_x\Gamma_1^*(x,-t,y,0).$$

Consequently, $adj(S_t^1\nabla_\|) = \nabla_\| S_{-t}^{L_1^*}$, so that L^2 boundedness of $(S_t^1\nabla)$ (and hence of $\mathcal{D}_t^1$) follows from that of $\nabla S_{-t}^{L_1^*}$. Thus, by Lemma 4.18, we also obtain L^2 bounds for $K^1, \widetilde{K}^1$ and $\mathcal{T}^1$. Appropriate non-tangential control follows from Lemma 4.8. Moreover, since we have allowed complex coefficients, analytic perturbation theory implies that

$$\|K^0 - K^1\|_{2\to 2} + \|\widetilde{K}^0 - \widetilde{K}^1\|_{2\to 2} + \|\mathcal{T}^0 - \mathcal{T}^1\|_{2\to 2} \leq C\|A^0 - A^1\|_\infty.$$

The method of continuity then yields the invertibility of $\pm\frac{1}{2}I + K^1 : L^2 \to L^2$, $\pm\frac{1}{2}I + \widetilde{K}^1 : L^2 \to L^2$ and $S_0^1 \equiv S_t^1|_{t=0} : L^2 \to \dot{L}_1^2$. It therefore suffices to prove (5.1).

Lemma 5.2. *Suppose that L, L^* satisfy the standard assumptions. For $f \in C_0^\infty$, $\eta > 0$, and $t_0 \geq 0$, we have*

$$\|\nabla_\| S_{t_0} f\|_2 \leq C\left(\|N_*(P_t\partial_t S_{t+t_0} f)\|_2 + |||t\nabla\partial_t S_t f||| + \|f\|_2\right) \tag{5.3}$$

$$\|\nabla_\| S_{t_0}^\eta f\|_2 \leq C\left(\|N_*\left(P_t\partial_t S_{t+t_0}^\eta f\right)\|_2 + |||t\nabla\partial_t S_t^\eta f||| + \|f\|_2\right) \tag{5.4}$$

$$|||t\nabla\partial_t S_t f||| \leq C|||t\partial_t^2 S_t f||| + C\|f\|_2 \tag{5.5}$$

$$|||t\nabla\partial_t S_t^\eta f||| \leq C|||t\partial_t^2 S_t^\eta f||| + C\|f\|_2. \tag{5.6}$$

The analogous bounds hold also in the lower half-space.

Before proving the lemma, let us use it to reduce the proof of Theorem 1.11 to two main estimates, whose proofs we shall give in the next two sections. We claim that it suffices to prove that for all $f \in C_0^\infty$, and $\eta \in (0, 10^{-10})$, we have

(5.7)

$$|||t\partial_t^2 S_t^{1,\eta} f|||_{all} \le C\epsilon_0 \left(|||t\nabla\partial_t S_t^{1,\eta} f|||_{all} + \|N_*^{db}\left(P_t\partial_t S_t^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\nabla S_t^{1,\eta} f\|_2 \right) + C\|f\|_2$$

(5.8)

$$\sup_{t\neq 0}\|\partial_t S_t^{1,\eta} f\|_2 \le C\epsilon_0 \left(|||t\nabla\partial_t S_t^{1,\eta} f|||_{all} + \|N_*^{db}\left(P_t\partial_t S_t^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\nabla S_t^{1,\eta} f\|_2 \right) + C\|f\|_2,$$

where N_*^{db} denotes the non-tangential maximal operator with respect to the double cone $\gamma^{db}(x) \equiv \gamma^+(x) \cup \gamma^-(x) \equiv \{(y,t) \in \mathbb{R}^{n+1} : |x-y| < |t|\}$. Indeed, for ϵ_0 sufficiently small, Lemma 2.18 (*iii*) and (5.6) allow us to hide the small triple bar norm in (5.7), so that

$$|||t\nabla\partial_t S_t^{1,\eta} f|||_{all} \le C\epsilon_0 \left(\|N_*^{db}\left(P_t\partial_t S_t^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\nabla S_t^{1,\eta} f\|_2 \right) + C\|f\|_2. \tag{5.9}$$

Using (5.4), (5.9) and hiding the small gradient term via Lemma 2.18 (*i*, *ii*), we obtain

$$\sup_{t\neq 0}\|\nabla S_t^{1,\eta} f\|_2 \le C\left(\sup_{t_0\ge 0}\|N_*^{db}\left(P_t\partial_t S_{t\pm t_0}^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\partial_t S_t^{1,\eta} f\|_2 + \|f\|_2 \right), \tag{5.10}$$

where the notation $N_*^{db}\left(P_t\partial_t S_{t\pm t_0}^{1,\eta} f\right)$ is interpreted to mean $t+t_0$ in the upper cone γ^+, and $t-t_0$ in the lower cone γ^-. Feeding the latter estimate back into (5.9), we obtain

$$|||t\nabla\partial_t S_t^{1,\eta} f|||_{all} \le C\epsilon_0 \left(\sup_{t_0\ge 0}\|N_*^{db}\left(P_t\partial_t S_{t\pm t_0}^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\partial_t S_t^{1,\eta} f\|_2 \right) + C\|f\|_2. \tag{5.11}$$

Combining (5.8), (5.10) and (5.11), we have

$$\sup_{t\neq 0}\|\partial_t S_t^{1,\eta} f\|_2 \;\le\; C\|f\|_2 + C\epsilon_0 \left(\sup_{t_0\ge 0}\|N_*^{db}\left(P_t\partial_t S_{t\pm t_0}^{1,\eta} f\right)\|_2 + \sup_{t\neq 0}\|\partial_t S_t^{1,\eta} f\|_2 \right).$$

Since $f \in C_0^\infty$, there is a large cube Q centered at 0 containing the support of f. By Lemma 4.8 (*iv*), taking a supremum over all $f \in C_0^\infty(Q)$, with $\|f\|_{L^2(Q)} = 1$, we have

$$\sup_{t\neq 0}\|\partial_t S_t^{1,\eta}\|_{L^2(Q)\to L^2(\mathbb{R}^n)} \;\le\; C\left(1 + \epsilon_0 \sup_{t\neq 0}\|\partial_t S_t^{1,\eta}\|_{L^2(Q)\to L^2(\mathbb{R}^n)}\right).$$

Using Lemma 2.18 (*vi*), we may hide the small term to obtain

$$\sup_{t\neq 0}\|\partial_t S_t^{1,\eta}\|_{L^2(Q)\to L^2(\mathbb{R}^n)} \;\le\; C \tag{5.12}$$

uniformly in Q. Thus, letting $\ell(Q) \to \infty$, and then $\eta \to 0$, we obtain by Lemma 2.18 (*iv*) that

$$\sup_{t\neq 0}\|\partial_t S_t^{1}\|_{2\to 2} \le C. \tag{5.13}$$

In addition, (5.12), Lemma 4.8 (*iv*) and a limiting argument as $\ell(Q) \to \infty$ imply that

$$\sup_{t_0\ge 0}\|N_*^{db}\left(P_t\partial_t S_{t\pm t_0}^{1,\eta} f\right)\|_2 \le C\|f\|_2, \quad f \in L^2(\mathbb{R}^n).$$

The latter estimate, (5.11), (5.12) and Lemma 2.18 (*v*) yield the bound for the first term in (5.1). The bound for the second term in (5.1) follows from (5.3), the bound just established for $|||t\nabla\partial_t S_t^1|||_{op}$, the fact that $N_*\left(P_t\partial_t S_{t+t_0} f\right) \le CM\left(N_*(\partial_t S_t f)\right)$, Lemma 4.8 (*i*) and (5.13).

The estimates (5.7) and (5.8) are the heart of the matter, and will be proved in sections 6 and 7, respectively.

We return now to the proof of the lemma.

Proof of Lemma 5.2. We prove (5.5) first. We have that $|||t\nabla\partial_t S_t f|||^2 =$

$$\lim_{\varepsilon\to 0}|||t\nabla\partial_t S_t f|||^2(\varepsilon) \equiv \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon}\nabla\partial_t S_t f\cdot\overline{\nabla\partial_t S_t f}\,tdt$$
$$= -\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\varepsilon}^{1/\varepsilon}\partial_t(\nabla\partial_t S_t f\cdot\overline{\nabla\partial_t S_t f})t^2dt + \text{“OK”},$$

where we may use Lemma 2.8(ii) to dominate the “OK” boundary terms by $C\|f\|_2^2$. By Cauchy's inequality, we then obtain that

$$|||t\nabla\partial_t S_t f|||^2(\varepsilon) \leq \delta\,|||t\nabla\partial_t S_t f|||^2(\varepsilon) + \frac{C}{\delta}|||t^2\nabla\partial_t^2 S_t f|||^2(\varepsilon) + C\|f\|_2^2,$$

where δ is at our disposal. For δ small, we can hide the first term. The second term is bounded by $|||t\partial_t^2 S_t f|||$, as may be seen by splitting $\mathbb{R}^{n+1}_+$ into Whitney boxes, and applying Caccioppoli's inequality. The bound (5.5) now follows.

The proof of (5.6) is similar. We write

$$|||t\nabla\partial_t S_t^{1,\eta} f|||^2 = \int_0^{2\eta}\int_{\mathbb{R}^n} + \int_{2\eta}^{\infty}\int_{\mathbb{R}^n} \equiv I + II.$$

Term II may be handled just like (5.5), since by definition (2.17),

$$|t\nabla\partial_t S_t^{\eta} f| \leq C\left(\varphi_\eta * \left(1_{s>\eta}|s\nabla\partial_s S_s f|\right)\right)(t), \quad t > 2\eta,$$

and $u(x,t) \equiv \partial_t^2 S_t^{\eta} f(x)$ solves $Lu = 0$ in the half space $\{t > \eta\}$. We omit the details. To bound term I, we note that by definition (2.17), $\partial_t S_t^{\eta} f(x) = L^{-1}(D_{n+1} f_\eta)(x,t)$, where $f_\eta(y,s) \equiv f(y)\varphi_\eta(s)$, so that

$$|I| \leq C\eta\iint|\nabla L^{-1}(D_{n+1}f_\eta)|^2dxdt \leq C\eta\left(\int|\varphi_\eta(t)|^2dt\right)\|f\|_2^2 = C\|f\|_2^2,$$

where we have used that $\nabla L^{-1}\,\mathrm{div} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$.

Next, we prove (5.3). By the ellipticity of the sub-matrix $A_{\|}$, we have that

$$\|\nabla_{\|}S_{t_0}f\|_2 \leq C\|A_{\|}\nabla_{\|}S_{t_0}f\|_2.$$

Now let $\vec{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, with $\|\vec{g}\|_2 = 1$. By the Hodge decomposition [AT, p. 116], we have that $\vec{g} = \nabla_x F + \vec{h}$, where $F \in \dot{L}_1^2(\mathbb{R}^n)$, $\|\nabla_x F\|_2 \leq C\|\vec{g}\|_2$ (C depending only on ellipticity), $h \in L^2(\mathbb{R}^n)$ and $\mathrm{div}_{\|}(A_{\|})^*\vec{h} = 0$ in the sense that $\int A_{\|}\nabla_{\|}\zeta\cdot\overline{\vec{h}} = 0$ for all $\zeta \in \dot{L}_1^2$. Lemma 2.9, with $m = -1$, ensures that $S_{t_0}f \in \dot{L}_1^2$, (albeit without quantitative bounds). Thus, for $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\langle A_{\|}\nabla_{\|}S_{t_0}f, \vec{g}\rangle = \langle A_{\|}\nabla_{\|}S_{t_0}f, \nabla_{\|}F\rangle,$$

and it suffices to bound the latter expression with $F \in C_0^\infty$. Now,

$$\langle A_{\|}\nabla_{\|}S_{t_0}f, \nabla_{\|}F\rangle = -\int_0^\infty \partial_t\langle A_{\|}\nabla_{\|}e^{-t^2L_{\|}}S_{t+t_0}f, \nabla_{\|}e^{-t^2(L_{\|})^*}F\rangle dt$$
$$= 2\int_0^\infty\left\{\langle A_{\|}\nabla_{\|}tL_{\|}e^{-t^2L_{\|}}S_{t+t_0}f, \nabla_{\|}e^{-t^2(L_{\|})^*}F\rangle + \langle A_{\|}\nabla_{\|}e^{-t^2L_{\|}}S_{t+t_0}f, \nabla_{\|}t(L_{\|})^*e^{-t^2(L_{\|})^*}F\rangle\right\}dt$$
$$- \int_0^\infty\langle A_{\|}\nabla_{\|}e^{-t^2L_{\|}}\partial_t S_{t+t_0}f, \nabla_{\|}e^{-t^2(L_{\|})^*}F\rangle dt = I + II - III.$$

Integrating by parts, we see that

$$
\begin{aligned}
(5.14)\quad |I+II| &= 4\left|\int_0^\infty\int_{\mathbb{R}^n}\left(L_\| e^{-t^2L_\|}S_{t+t_0}f(x)\right)\left(\overline{(L_\|)^* e^{-t^2(L_\|)^*}F(x)}\right)tdxdt\right| \\
&\le 4|||te^{-t^2L_\|}L_\| S_{t+t_0}f|||\,|||t(L_\|)^*e^{-t^2(L_\|)^*}F||| \le C|||te^{-t^2L_\|}L_\| S_{t+t_0}f|||\,\|\nabla F\|_2,
\end{aligned}
$$

since, by [AHLMcT], applied to $(L_\|)^*$, we have that $|||t(L_\|)^*e^{-t^2(L_\|)^*}F||| \le C\|\nabla F\|_2$. We consider now the first factor on the right side of (5.14). Since $u(x,t) \equiv S_{t+t_0}f(x)$ solves $Lu = 0$, we have

$$
L_\| S_{t+t_0}f = \sum_{i=1}^{n} D_iA_{i,n+1}D_{n+1}S_{t+t_0}f + \sum_{j=1}^{n+1} A_{n+1,j}D_jD_{n+1}S_{t+t_0}f \equiv \Sigma_1+\Sigma_2,
$$

in the weak sense of Lemma 2.9. Since $e^{-t^2L_\|}: L^2\to L^2$ uniformly in t, we obtain

$$
|||te^{-t^2L_\|}\Sigma_2||| \le C|||t\nabla\partial_t S_{t+t_0}f||| \le C|||t\nabla\partial_t S_t f|||
$$

which is one of the allowable terms in the bound that we seek. Also,

$$
(5.15)\qquad te^{-t^2L_\|}\Sigma_1 = R_t\partial_t S_{t+t_0}f + \sum_{i=1}^{n}(te^{-t^2L_\|}D_iA_{i,n+1})P_t\partial_t S_{t+t_0}f,
$$

where, by the familiar “Gaffney estimate”(e.g., [AHLMcT], pp. 636-637), the operator

$$
R_t \equiv \sum_{i=1}^{n}\left(te^{-t^2L_\|}D_iA_{i,n+1} - (te^{-t^2L_\|}D_iA_{i,n+1})P_t\right)
$$

satisfies the bound (3.1) for every $m \ge 1$ (indeed, it satisfies a stronger exponential decay estimate). Moreover, $R_t 1 = 0$, and $R_t: L^2 \to L^2$. Thus, by Lemma 3.5 we have

$$
|||R_t\partial_t S_{t+t_0}f||| \le C|||t\nabla\partial_t S_{t+t_0}f||| \le C|||t\nabla\partial_t S_t f|||
$$

as desired. In addition, by [AHLMcT], we have that $|te^{-t^2L_\|}\operatorname{div}_\| \vec{b}|^2\frac{dxdt}{t}$ is a Carleson measure for all $\vec{b}\in L^\infty(\mathbb{R}^n,\mathbb{C}^n)$. Therefore, by Carleson's Lemma, the triple bar norm of the last term in (5.15) is dominated by $\|N_*(P_t\partial_t S_{t+t_0}f)\|_2$.

It remains to handle the term III. Integrating by parts in t, we obtain

$$
(5.16)\qquad -III = \int_0^\infty \langle A_\|\nabla_\| e^{-t^2L_\|}\partial_t^2 S_{t+t_0}f, \nabla_\| e^{-t^2(L_\|)^*}F\rangle tdt \; + \; \text{“easy”},
$$

where the two “easy terms” arise when ∂_t hits either $e^{-t^2L_\|}$ or $e^{-t^2(L_\|)^*}$. These two easy terms may be handled by an argument similar to, but simpler than the one used to treat (5.14) above. The main term in (5.16) is dominated by

$$
|||te^{-t^2L_\|}\partial_t^2 S_{t+t_0}f|||\;|||t(L_\|)^*e^{-t^2(L_\|)^*}F||| \le C|||t\partial_t^2 S_t f|||\;\|\nabla F\|_2,
$$

where we have used the L^2 boundedness of $e^{-t^2L_\|}$ to estimate the first factor, and [AHLMcT] to handle the second.

Finally, (5.4) may be proved in the same way as (5.3) with one minor modification. Since $LS_t^\eta f(x) = f_\eta(x,t) \equiv f(x)\varphi_\eta(t)$, the application of Lemma 2.9 produces, in addition to the analogues of Σ_1 and Σ_2, an error term $f_\eta(\cdot, t+t_0)$. But

$$
|||te^{-t^2L_\|}f_\eta(\cdot,t+t_0)||| \le C\left(\eta\int|\varphi_\eta(t+t_0)|^2dt\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)} = C\|f\|_2,
$$

and (5.4) follows. □

We finish this section with a variant of the square function estimates.

Lemma 5.17. *Suppose that L, L^* satisfy the standard assumptions, and have "Good Layer Potentials". Then for $m \geq 0$, we have the square function bound*

$$|||t^{m+1}\partial_t^{m+1}(S_t\nabla)\cdot \mathbf{f}||| \leq C_m\|\mathbf{f}\|_2,$$

where $\mathbf{f} \in L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$.

Proof. By t-independence and Caccioppoli's inequality in Whitney boxes, we may reduce to the case $m = 0$. By t-independence and (1.10), we may replace ∇ by $\nabla_{\|}$. By ellipticity of the $n \times n$ sub-matrix $A_{\|}$, and the Hodge decomposition of [AT, p. 116], as in the proof of Lemma 5.2, it suffices to show that

$$|||t\partial_t(S_t\nabla_{\|})\cdot A_{\|}\nabla_{\|}F||| \leq C\|\nabla_{\|}F\|_2, \tag{5.18}$$

with $F \in \{S_0\psi : \psi \in C_0^\infty\}$ (which is dense in $\dot{L}_1^2$, by the bijectivity of the mapping $S_0 : L^2 \to \dot{L}_1^2$). In the weak sense of Lemma 2.9, we have

$$\overline{(L_{\|})_y^*}\Gamma(x,t,y,s) = \sum_{i=1}^n \frac{\partial}{\partial y_i}\left(\overline{A^*_{i,n+1}(y)}\partial_s\Gamma(x,t,y,s)\right) + \sum_{j=1}^{n+1}\overline{A^*_{n+1,j}(y)}\frac{\partial}{\partial y_j}\partial_s\Gamma(x,t,y,s).$$

By t-independence, we therefore have that

$$\partial_t(S_t\nabla_{\|})\cdot A_{\|}\nabla_{\|}F = \sum_{i=1}^n \partial_t^2 S_t A_{n+1,i}D_iF + \partial_t^2(S_t\overline{\partial_{\nu^*}})F,$$

where $\overline{\partial_{\nu^*}} = -\sum_{j=1}^{n+1}\overline{A^*_{n+1,j}}D_j$. We set $u(\cdot,\tau) = S_\tau\psi, \tau < 0$, so that $u(\cdot,0) \equiv F$. Using "Good Layer Potentials", we obtain in particular that

$$\|\nabla u(\cdot,0)\|_2 \leq C\|\nabla_{\|}F\|_2. \tag{5.19}$$

Since $(S_t\partial_{\nu^*}) = \mathcal{D}_t$, Corollary 4.28 implies that

$$\partial_t^2(S_t\overline{\partial_{\nu^*}})F = \partial_t^2 S_t(\partial_\nu u(\cdot,0)).$$

Consequently, the left hand side of (5.18) is dominated by

$$\sum_{i=1}^n |||t\partial_t^2 S_t A_{n+1,i}D_iF||| + |||t\partial_t^2 S_t(\partial_\nu u(\cdot,0))||| \leq C\|\nabla_{\|}F\|_2,$$

where in the last step we have used (1.10) and (5.19). $\square$

6. P T 1.11: (5.7)

In this section we prove estimate (5.7). To be precise, suppose that $\varphi_\delta = \delta^{-1}\varphi(\cdot/\delta)$ is the kernel of a nice approximate identity in 1 dimension, as in the definition of S_t^η (2.17). We shall prove that, for all $f \in C_0^\infty(\mathbb{R}^n)$, for all $\Psi \in C_0^\infty(\mathbb{R}^{n+1}_+)$, with $|||\Psi||| \leq 1$, and for all $\delta > 0$ sufficiently small, if $\Psi_\delta(x,t) \equiv \varphi_\delta * \Psi(x,\cdot)(t)$, then

$$\iint_{\mathbb{R}^{n+1}_+} t\partial_t^2 S_t^{1,\eta}f(x)\,\overline{\Psi_\delta(x,t)}\,\frac{dx\,dt}{t} \leq C\epsilon_0\left(\mathbf{M}^+ + \mathbf{M}^-\right) + C\|f\|_2, \tag{6.1}$$

where

$$\mathbf{M}^+ \equiv \left(|||t\nabla\partial_t S_t^{1,\eta}f|||_+ + \|N_*\left(P_t\partial_t S_t^{1,\eta}f\right)\|_2 + \sup_{t\geq 0}\|\nabla S_t^{1,\eta}f\|_2 + \|f\|_2\right), \tag{6.2}$$

and $\mathbf{M}^-$ is the corresponding quantity for the lower half-space. The proof of the analogous estimate in $\mathbb{R}^{n+1}_-$ is identical, and we omit it. By Lemma 2.18 (*iii*), we may take first the limit as $\delta \to 0$, and then the supremum over all such Ψ to obtain (5.7).

The proof is by perturbation. Setting $\epsilon(z) \equiv A^1(z) - A^0(z)$, we have

$$L_0^{-1} - L_1^{-1} = L_0^{-1} L_1 L_1^{-1} - L_0^{-1} L_0 L_1^{-1} = -L_0^{-1} \operatorname{div} \epsilon \nabla L_1^{-1}.$$

Since $|||t\partial_t^2 S_t^0 f||| \leq C\|f\|_2$, we have also that $\sup_{\eta>0} |||t\partial_t^2 S_t^{0,\eta} f||| \leq C\|f\|_2$, as may be seen by arguing as in the proof of (5.6). Thus, it is enough to consider the difference $t\partial_t^2 \left(S_t^{1,\eta} - S_t^{0,\eta}\right)$. By definition (2.17),

$$\partial_t S_t^{i,\eta} f(x) = \left((D_{n+1}\varphi_\eta) * S_{(\cdot)}^i f(x)\right)(t) = L_i^{-1}(D_{n+1} f_\eta)(x,t), \quad i = 1, 2, \tag{6.3}$$

where $f_\eta(y,s) \equiv f(y)\varphi_\eta(s)$, and $\varphi_\eta = \eta^{-1}\varphi(\cdot/\eta)$ is as above. We then have

$$\begin{aligned} \partial_t^2 S_t^{1,\eta} f(x) - \partial_t^2 S_t^{0,\eta} f(x) &= \partial_t \left(L_0^{-1} \operatorname{div} \epsilon \nabla L_1^{-1}(D_{n+1} f_\eta)\right)(x,t) \\ &= \partial_t \left(L_0^{-1} \operatorname{div} \epsilon \nabla D_{n+1} S_{(\cdot)}^{1,\eta} f\right)(x,t), \end{aligned}$$

so that

$$\iint_{\mathbb{R}^{n+1}_+} \left(t\partial_t^2 S_t^{1,\eta} f(x) - t\partial_t^2 S_t^{0,\eta} f(x)\right) \overline{\Psi_\delta(x,t)}\, \frac{dx\,dt}{t} =$$

$$\iint_{\mathbb{R}^{n+1}} \epsilon(y) \nabla\partial_s S_s^{1,\eta} f(y) \cdot \overline{\nabla(L_0^*)^{-1}(D_{n+1}\Psi_\delta)(y,s)}\, dyds. \tag{6.4}$$

Essentially following [FJK], and using (6.3), we decompose

$$\begin{aligned} \nabla(L_0^*)^{-1}(D_{n+1}\Psi_\delta)(y,s) &= \int \nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\left(\Psi(\cdot,t)\right)(y)\, dt \\ &= \int_{t>2|s|} \left\{ \nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\left(\Psi(\cdot,t)\right)(y) - \left(\nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\right)\Big|_{s=0}\left(\Psi(\cdot,t)\right)(y) \right\} dt \\ &\quad + \int_{t>2|s|} \left(\nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\right)\Big|_{s=0}\left(\Psi(\cdot,t)\right)(y)\, dt \\ &\quad + \int_{t\leq 2|s|} \left(\frac{\sqrt{t} - \sqrt{|s|}}{\sqrt{t}}\right) \nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\left(\Psi(\cdot,t)\right)(y)\, dt \\ &\quad + \int \left(\frac{|s|}{t}\right)^{1/2} \nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\left(\Psi(\cdot,t)\right)(y)\, dt \\ &\quad - \int_{t>2|s|} \left(\frac{|s|}{t}\right)^{1/2} \nabla_{y,s}\partial_s S_{s-t}^{L_0^*,\delta}\left(\Psi(\cdot,t)\right)(y)\, dt \;\equiv\; \mathbf{i} + \mathbf{ii} + \mathbf{iii} + \mathbf{iv} - \mathbf{v}. \end{aligned}$$

In turn, this induces a corresponding decomposition in (6.4):

$$I + II + III + IV - V \equiv \iint_{\mathbb{R}^{n+1}} \epsilon(y)\nabla\partial_s S_s^{1,\eta} f(y) \cdot \overline{(\mathbf{i} + \mathbf{ii} + \mathbf{iii} + \mathbf{iv} - \mathbf{v})}\, dyds.$$

All but term II will be easy to handle, and we shall deal with these easy terms as in [FJK]. The main term here (and in [FJK]) is II, but in our situation, matters are much more delicate, since for us A^0 is not constant. The approach of [FJK] depends critically on the fact that solutions of constant coefficient equations are, in particular, twice differentiable, a fact which fails utterly in the present setting (unless at least one of the derivatives falls on the t-variable). We shall require new methods, which exploit the technology of the solution of the Kato problem, to deal with term II.

We dispose of the easy terms in short order. To begin,

$$IV = \iint_{\mathbb{R}^{n+1}} |s|^{1/2}\epsilon(y)\nabla\partial_s S_s^{1,\eta} f(y) \cdot \nabla(L_0^*)^{-1}\left(D_{n+1}\left(\varphi_\delta * \frac{\Psi}{\sqrt{t}}\right)\right)(y,s)\, dyds.$$

Since $\nabla L_0^{-1}\operatorname{div} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$, we have that $|IV| \le C\epsilon_0 |||t\nabla\partial_t S_t^{1,\eta} f|||_{all}$. Given the following lemma, I, III and V may be handled by Hardy's inequality, yielding also the bound $|I| + |III| + |V| \le C\epsilon_0 |||t\nabla\partial_t S_t^{1,\eta} f|||_{all}$. We omit the details.

Lemma 6.5. *We have*

$$\|\nabla D_{n+1} S_{s-t}^{L_0^*,\delta} - \nabla D_{n+1} S_{-t}^{L_0^*,\delta}\|_{2\to 2} \le C\frac{|s|}{t^2}, \quad |s| < t/2,\ \delta < 1000^{-1} t \tag{6.6}$$

$$\|\nabla\partial_\tau S_\tau^{L_0^*,\delta}\|_{2\to 2} \le \frac{C}{|\tau|}, \quad \tau \ne 0 \tag{6.7}$$

Proof of the Lemma. If $|\tau| > 100\delta$, estimate (6.7) is essentially just the case $m = 0$ of Lemma 2.8. Otherwise, we obtain the better bound $C\delta^{-1}$, using definition (2.17) and the hypothesis that L_0, L_0^* have bounded layer potentials. Estimate (6.6) is obtained from the case $m = 1$ of Lemma 2.8, and the identity

$$\nabla D_{n+1} S_{s-t}^{L_0^*,\delta} - \nabla D_{n+1} S_{-t}^{L_0^*,\delta} = \int_0^s \nabla\partial_\tau^2 S_{\tau-t}^{L_0^*,\delta}\, d\tau.$$

□

It remains to handle II, which equals

$$\iint_{\mathbb{R}^{n+1}} \left\{\int_{-t/2}^{t/2} \epsilon(y)\nabla\partial_s S_s^{1,\eta} f(y)\, ds\right\} \cdot \overline{\left(\nabla D_{n+1} S_{-t}^{L_0^*,\delta}\right)(\Psi(\cdot,t))(y)}\, dydt$$

$$= -\iint_{\mathbb{R}^{n+1}_+} \left(\partial_t S_t^0 \nabla\right) \cdot \epsilon\nabla\left(S_{t/2}^{1,\eta} f - S_{-t/2}^{1,\eta} f\right)(x)\, \overline{\Psi_\delta(x,t)}\, dxdt, \tag{6.8}$$

where we have used that for $\eta > 0$, ∇S_t^η does not jump across the boundary. Since Ψ is compactly supported in R^{n+1}_+, for δ sufficiently small,

$$t^{-1/2}|\Psi_\delta(x,t)| \le C \int \varphi_\delta(t-s)|\Psi(x,s)| s^{-1/2} ds.$$

Thus, it is enough to bound $|||t\left(\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_{t/2}^{1,\eta} f|||$, plus a similar term with $-t/2$ in place of $t/2$, which may be handled in the same way. The desired bound then follows immediately from the change of variable $t \to 2t$ and (6.10) below.

Lemma 6.9. *Suppose that $a \in \mathbb{R}\setminus\{0\}$, and define $\mathbf{M}^+$ as in* (6.2)*. Then*

$$|||t\left(\partial_t S_{at}^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f||| \quad\le\quad C(a)\epsilon_0\,\mathbf{M}^+ \tag{6.10}$$

$$|||t^2\left(\partial_t^2 S_{at}^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f||| \quad\le\quad C(a)\epsilon_0\,\mathbf{M}^+. \tag{6.11}$$

Moreover, the analogous bound holds in the lower half space.

Proof of Lemma 6.9. This lemma is the deep fact underlying estimate (5.7), and the proof is rather delicate. For the sake of notational simplicity, we treat only the case $a = 1$, as the general case is handled by an almost identical argument. We begin by showing that (6.11) implies (6.10). Set

$$\mathbf{J}(\sigma) \equiv \int_\sigma^{1/\sigma}\int_{\mathbb{R}^n} \left|\partial_t\left(S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f\right|^2 dx\, tdt.$$

After integrating by parts in t, we obtain that

$$\mathbf{J}(\sigma) = -\Re e\int_\sigma^{1/\sigma}\int_{\mathbb{R}^n} \frac{\partial}{\partial t}\left\{\left(\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f\right\}\overline{\left\{\left(\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f\right\}}\, dx\, t^2 dt \ +\ \text{“OK”},$$

where by Lemma 2.8 (i), the "OK" boundary terms are dominated by $C\epsilon_0^2 \sup_{t>0} \|\nabla S_t^{1,\eta} f\|_2^2$. By Cauchy's inequality, modulo the "OK" terms,

$$\begin{aligned}\mathbf{J}(\sigma) \;&\leq\; \frac{1}{2}\mathbf{J}(\sigma) + |||t\left(\partial_t S_t^0 \nabla\right) \cdot \epsilon t \nabla \partial_t S_t^{1,\eta} f|||^2 + |||t^2\left(\partial_t^2 S_t^0 \nabla\right) \cdot \epsilon \nabla S_t^{1,\eta} f|||^2 \\ &\equiv \frac{1}{2}\mathbf{J}(\sigma) + I + II.\end{aligned}$$

The term $\frac{1}{2}\mathbf{J}(\sigma)$ may be hidden on the left hand side. By Lemma 2.8 (i) with $m = 0$, term I is no larger than $C\epsilon_0^2 |||t\nabla\partial_t S_t^{1,\eta} f|||^2$. The square root of the main term, II, is estimated in (6.11). Taking the latter for granted momentarily, we obtain (6.10) by letting $\sigma \to 0$.

We now turn to the proof of (6.11), again with $a = 1$. We make the splitting:

$$\begin{aligned}t^2\partial_t^2(S_t^0\nabla) \cdot \epsilon \nabla S_t^{1,\eta} f \;&=\; \sum_{i=1}^{n+1}\sum_{j=1}^{n} t^2\partial_t^2(S_t^0 D_i)\epsilon_{ij} D_j S_t^{1,\eta} f \\ &\quad + \sum_{i=1}^{n+1} t^2\partial_t^2(S_t^0 D_i)\epsilon_{i,n+1} D_{n+1} S_t^{1,\eta} f \;\equiv\; V_t f + \widetilde{V}_t f.\end{aligned}$$

We treat $\widetilde{V}_t$ first. For $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$, set

$$\theta_t \mathbf{f} \equiv t^2\partial_t^2(S_t^0\nabla) \cdot \mathbf{f},$$

and let $\vec{\epsilon} \equiv (\epsilon_{1,n+1}, \epsilon_{2,n+1}, \ldots, \epsilon_{n+1,n+1})$. Then, using a well known trick of [CM], we write

$$\widetilde{V}_t f = \{\theta_t\vec{\epsilon} - (\theta_t\vec{\epsilon})P_t\}\,\partial_t S_t^{1,\eta} f \;+\; (\theta_t\vec{\epsilon})P_t\partial_t S_t^{1,\eta} f \;\equiv\; R_t^\epsilon \partial_t S_t^{1,\eta} f + (\theta_t\vec{\epsilon})P_t\partial_t S_t^{1,\eta} f,$$

where as usual P_t is a nice approximate identity. By Lemmas 5.17, 2.7, 3.2 and Carleson's Lemma, the triple bar norm of the second summand is no larger than $C\epsilon_0\|N_*\left(P_t\partial_t S_t^{1,\eta} f\right)\|_2$. In addition, by Lemma 3.5, we have that

$$|||R_t^\epsilon \partial_t S_t^{1,\eta} f||| \;\leq\; C\epsilon_0 |||t\nabla_\| \partial_t S_t^{1,\eta} f||| \;\leq\; C\epsilon_0 |||t\nabla\partial_t S_t^{1,\eta} f|||.$$

It remains to control $|||V_t f|||$, which is the primary difficulty. By definition,

$$V_t \;=\; \theta_t \tilde{\epsilon}\nabla_\| S_t^{1,\eta} \;\equiv\; t^2\partial_t^2(S_t^0\nabla) \cdot \tilde{\epsilon}\nabla_\| S_t^{1,\eta},$$

where $\tilde{\epsilon}$ is the $(n+1) \times n$ matrix $(\epsilon_{ij})_{1\leq i\leq n+1, 1\leq j\leq n}$. Recall that $A_\|^1$ is the $n \times n$ sub-matrix of A^1 with $(A_\|^1)_{ij} = A_{ij}^1$, $1 \leq i, j \leq n$, and that $(L_1)_\| \equiv -\operatorname{div}_\| A_\|^1 \nabla_\|$. Then

$$V_t \;=\; \theta_t\tilde{\epsilon}\nabla_\|\left(I - \left(I + t^2(L_1)_\|\right)^{-1}\right)S_t^{1,\eta} \;+\; \theta_t\tilde{\epsilon}\,\nabla_\|(I + t^2(L_1)_\|)^{-1} S_t^{1,\eta} \;\equiv\; Y_t + Z_t.$$

We first consider Y_t. Note that $\left(I - \left(I + t^2(L_1)_\|\right)^{-1}\right) = t^2(L_1)_\| \left(I + t^2(L_1)_\|\right)^{-1}$, so

$$Y_t = \theta_t\tilde{\epsilon}t^2\nabla_\|\left(I + t^2(L_1)_\|\right)^{-1}(L_1)_\| S_t^{1,\eta}.$$

As above, set $f_\eta(x,t) \equiv f(x)\varphi_\eta(t)$. In the weak sense of Lemma 2.9, we then have

$$(L_1)_\| S_t^{1,\eta} f = \sum_{i=1}^{n} D_i A_{i,n+1}^1 \partial_t S_t^{1,\eta} f + \sum_{j=1}^{n+1} A_{n+1,j}^1 D_j \partial_t S_t^{1,\eta} f + f_\eta,$$

and we denote by $Y_t^{(1)} + Y_t^{(2)} + Y_t^{(3)}$ the corresponding splitting of Y_t. Now, by Lemma 2.8, $\theta_t : L^2 \to L^2$, and it is well known that $t\nabla_\|(I + t^2(L_1)_\|)^{-1} : L^2 \to L^2$. Thus

$$|||Y_t^{(2)} f||| \leq C\epsilon_0 |||t\nabla\partial_t S_t^{1,\eta} f|||,$$

and also, as in the proof of (5.6),

$$|||Y_t^{(3)}||| \leq C\epsilon_0 |||t f_\eta||| \leq C\epsilon_0 \|f\|_{L^2(\mathbb{R}^n)}.$$

We make a further decomposition of $Y_t^{(1)}$ as follows:

$$Y_t^{(1)} = (U_t\vec{a} - (U_t\vec{a})P_t)\,\partial_t S_t^{1,\eta} + (U_t\vec{a})P_t\partial_t S_t^{1,\eta} \;\equiv\; \widetilde{R}_t\partial_t S_t^{1,\eta} + (U_t\vec{a})P_t\partial_t S_t^{1,\eta}\,,$$

where

$$U_t\vec{g} \;\equiv\; \theta_t\tilde{\epsilon}t^2\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} \mathrm{div}_\| \,\vec{g}', \tag{6.12}$$

and $\vec{a} \equiv (A^1_{1,n+1}, A^1_{2,n+1}, \ldots, A^1_{n,n+1})$. We now claim that

$$|||U_t|||_{op} \leq C\epsilon_0 \tag{6.13}$$

Let us momentarily defer the proof of this claim. It is a standard fact that for two sets E and $E' \subseteq \mathbb{R}^n$, with $\vec{g}$ supported in E', we have

$$\left\|t^2\nabla_\| \left(1 + t^2(L_1)_\|\right)^{-1} \mathrm{div}_\| \,\vec{g}\right\|_{L^2(E)} \leq C\exp\left\{\frac{-\operatorname{dist}(E, E')}{Ct}\right\}\|\vec{g}\|_{L^2(E')} \tag{6.14}$$

(the corresponding fact for the operator $t\nabla_\| \left(1 + t^2(L_1)_\|\right)^{-1}$ is proved in [AHLMcT] for example, and (6.14) may be readily deduced from this fact plus the same argument). Thus, by Lemma 3.3, the operator U_t satisfies (3.1), with a bound on the order of $C\epsilon_0$, whenever $t \leq c\ell(Q)$. Therefore, by Lemma 3.2 and Carleson's Lemma, we have that

$$|||(U_t\vec{a})P_t\partial_t S_t^{1,\eta} f||| \leq C\epsilon_0 \|N_*(P_t\partial_t S_t^{1,\eta} f)\|_2.$$

Moreover, by Lemmas 3.5 and 3.11, we have that

$$|||\widetilde{R}_t\partial_t S_t^{1,\eta} f||| \leq C\epsilon_0 |||t\nabla_\|\partial_t S_t^{1,\eta} f||| \leq C\epsilon_0 |||t\nabla\partial_t S_t^{1,\eta} f|||.$$

To finish our treatment of Y_t, it remains to prove (6.13). We continue to defer the proof of this estimate for the moment, and proceed to discuss the term Z_t. We write

$$\begin{aligned} Z_t \;=\; &\theta_t\tilde{\epsilon}\,\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} (S_t^{1,\eta} - S_0^{1,\eta}) \\ &+\; \theta_t\tilde{\epsilon}\,\nabla_\| \left(\left(I + t^2(L_1)_\|\right)^{-1} - I\right) S_0^{1,\eta} \;+\; \theta_t\tilde{\epsilon}\,\nabla_\| S_0^{1,\eta} \;\equiv\; Z_t^{(1)} + Z_t^{(2)} + Z_t^{(3)}. \end{aligned}$$

By Lemma 5.17 with $m = 1$, we have that

$$|||Z_t^{(3)} f||| \leq C\epsilon_0 \sup_{t>0} \|\nabla S_t^{1,\eta} f\|_2.$$

Also,

$$Z_t^{(2)} = \theta_t\tilde{\epsilon}\,\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} t^2 \,\mathrm{div}_\| \,A^1_\|\nabla_\| S_0^{1,\eta} \equiv U_t A^1_\| \nabla_\| S_0^{1,\eta}$$

(see (6.12)), so by the deferred estimate (6.13) we have that

$$|||Z_t^{(2)} f||| \leq C\epsilon_0 \sup_{t>0} \|\nabla S_t^{1,\eta} f\|_2.$$

Integrating by parts, we obtain

$$\begin{aligned} Z_t^{(1)} = \theta_t\tilde{\epsilon}\,\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} \int_0^t \partial_s S_s^{1,\eta} ds \;=\; &-\theta_t\tilde{\epsilon}\,\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} \int_0^t s\partial_s^2 S_s^{1,\eta} ds \\ &+\; \theta_t\tilde{\epsilon}\,\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1} t\partial_t S_t^{1,\eta} \;\equiv\; \Omega_t^{(1)} + \Omega_t^{(2)}. \end{aligned}$$

By Lemma 3.3, and the fact that $\nabla_\|(I + t^2(L_1)_\|)^{-1}1 = 0$, we have that the operator

$$R_t \equiv \theta_t\tilde{\epsilon}\, t\nabla_\| \left(I + t^2(L_1)_\|\right)^{-1}$$

satisfies the hypothesis of Lemma 3.5, with a bound on the order of $C\epsilon_0$, so that

$$|||\Omega_t^{(2)} f||| \leq C\epsilon_0\, |||t\nabla\partial_t S_t^{1,\eta} f|||.$$

Furthermore,

$$\Omega_t^{(1)} = -\int_0^t \frac{s}{t}\,\theta_t\tilde{\epsilon}\, t\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1} s\partial_s^2 S_s^{1,\eta}\frac{ds}{s},$$

so by Lemma 3.12, we have

$$|||\Omega_t^{(1)} f||| \leq C\epsilon_0\, |||t\partial_t^2 S_t^{1,\eta} f|||.$$

Modulo (6.13), this concludes the proof of Lemma 6.9, and hence also that of (5.7).

We conclude the present section by proving (6.13). The proof will depend on some technology from the proof of the Kato square root conjecture. By ellipticity, it is enough to show that

$$|||U_t A_{\|}^1 \vec{g}||| \leq C\epsilon_0 \|\vec{g}\|_2$$

for $\vec{g} \in L^2(\mathbb{R}^n, \mathbb{C}^n)$. But

$$U_t A_{\|}^1 = \theta_t\tilde{\epsilon}\, t^2\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1} \operatorname{div}_{\|} A_{\|}^1,$$

so by the Hodge decomposition [AT, p. 116], we may replace $\vec{g}$ by $\nabla_{\|}F$, where $\|\nabla_{\|}F\|_2 \leq C\|g\|_2$. As usual, by density we may suppose that $F \in C_0^\infty$. Now

$$U_t A_{\|}^1 \nabla_{\|} F \;=\; -\theta_t\tilde{\epsilon}\,\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1}(t^2(L_1)_{\|})F \;=\; \theta_t\tilde{\epsilon}\,\nabla_{\|}\left(\left(I + t^2(L_1)_{\|}\right)^{-1} - I\right)F.$$

We recall that $\theta_t = t^2\partial_t^2(S_t^0\nabla)\,\cdot$, so by Lemma 5.17 with $m = 1$,

$$|||\theta_t\tilde{\epsilon}\,\nabla_{\|}F||| \leq C\epsilon_0\|\nabla_{\|}F\|_2.$$

The main term is

$$\theta_t\tilde{\epsilon}\,\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1} F \equiv \frac{1}{t}R_t F,$$

where by Lemmas 2.7, 2.8, and 3.3, and the fact that $\nabla_{\|}(I + t^2L_2')^{-1}1 = 0$, we have that R_t satisfies the hypotheses of Lemma 3.9, with a bound on the order of $C\epsilon_0$. Therefore, it suffices to prove the Carleson measure estimate

$$\int_0^{\ell(Q)}\int_Q \left|\frac{1}{t}R_t\Phi(x)\right|^2 \frac{dxdt}{t} \leq C\epsilon_0|Q|,$$

where $\Phi(x) \equiv x$. To this end, we write

$$\frac{1}{t}R_t\Phi = \theta_t\tilde{\epsilon}\,\nabla_{\|}\left(\left(I + t^2(L_1)_{\|}\right)^{-1} - I\right)\Phi + \theta_t\tilde{\epsilon}\,\nabla_{\|}\Phi. \tag{6.15}$$

But $\nabla_{\|}\Phi = \mathbb{I}$, the $n \times n$ identity matrix. Thus, Lemmas 5.17, 2.7 and 3.2 yield the bound

$$\int_0^{\ell(Q)}\int_Q |\theta_t\tilde{\epsilon}\nabla_{\|}\Phi|^2\frac{dxdt}{t} \leq C\epsilon_0|Q|.$$

The remaining term in (6.15) equals

$$\theta_t\tilde{\epsilon}\, t^2\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1}\operatorname{div}_{\|} A_{\|}\nabla_{\|}\Phi \;=\; \theta_t\tilde{\epsilon}\, t^2\nabla_{\|}\left(I + t^2(L_1)_{\|}\right)^{-1}\operatorname{div}_{\|} A_{\|} \;\equiv\; T_t A_{\|}$$

We now invoke a key fact in the proof of the Kato conjecture. By [AHLMcT], there exists, for each Q, a mapping $F_Q = \mathbb{R}^n \to \mathbb{C}^n$ such that

$$
\begin{aligned}
&\text{(i)} \quad \int_{\mathbb{R}^n} |\nabla_{\|} F_Q|^2 \leq C|Q| \\
&\text{(ii)} \quad \int_{\mathbb{R}^n} |(L_1)_{\|} F_Q|^2 \leq C \frac{|Q|}{\ell(Q)^2} \\
&\text{(iii)} \quad \sup_Q \int_0^{\ell(Q)} \fint_Q |\vec{\zeta}(x,t)|^2 \frac{dxdt}{t} \\
&\qquad \leq C \sup_Q \int_0^{\ell(Q)} \fint_Q |\vec{\zeta}(x,t) E_t \nabla_{\|} F_Q(x)|^2 \frac{dxdt}{t},
\end{aligned}
\tag{6.16}
$$

for every function $\vec{\zeta} : \mathbb{R}^{n+1}_+ \to \mathbb{C}^n$, where E_t denotes the dyadic averaging operator, i.e. if $Q(x,t)$ is the minimal dyadic cube (with respect to the grid induced by Q) containing x, with side length at least t, then

$$E_t g(x) \equiv \fint_{Q(x,t)} g.$$

Here $\nabla_{\|} F_Q$ is the Jacobian matrix $(D_i(F_Q)_j)_{1\leq i,j\leq n}$, and the product

$$\vec{\zeta} E_t \nabla_{\|} F_Q = \sum_{i=1}^n \zeta_i E_t D_i F_Q$$

is a vector. Given the existence of a family of mappings F_Q with these properties, as in [AT, Chapter 3], we see by (iii), applied with $\vec{\zeta}(x,t) = T_t A_{\|}$, that it is enough to show that

$$\int_0^{\ell(Q)} \int_Q |T_t A_{\|}(x)\, E_t \nabla_{\|} F_Q(x)|^2 \frac{dxdt}{t} \leq C\epsilon_0 |Q|.$$

But as in [AT], we may exploit the idea of [CM] to write

$$
\begin{aligned}
(T_t A_{\|}) E_t \nabla_{\|} F_Q &= \{(T_t A_{\|}) E_t - T_t A_{\|}\} \nabla_{\|} F_Q + T_t A_{\|} \nabla_{\|} F_Q \\
&= (T_t A_{\|})(E_t - E_t P_t) \nabla_{\|} F_Q + \{(T_t A_{\|}) E_t P_t - T_t A_{\|}\} \nabla_{\|} F_Q + T_t A_{\|} \nabla_{\|} F_Q \\
&\equiv R_t^{(1)} \nabla_{\|} F_Q + R_t^{(2)} \nabla_{\|} F_Q + T_t A_{\|} \nabla_{\|} F_Q,
\end{aligned}
$$

where P_t is a nice approximate identify. The last term is easy to handle. We have that

$$T_t A_{\|} \nabla_{\|} F_Q = -\theta_t \tilde{\epsilon} t \nabla_{\|} \left(I + t^2 (L_1)_{\|} \right)^{-1} t (L_1)_{\|} F_Q.$$

Therefore, since θ_t and $t\nabla_{\|}(I + t^2(L_1)_{\|})^{-1}$ are uniformly bounded on L^2, we obtain that

$$\int_0^{\ell(Q)} \int_Q |T_t A_{\|} \nabla_{\|} F_Q|^2 \frac{dxdt}{t} \leq C\epsilon_0 \int_{\mathbb{R}^n} |(L_1)_{\|} F_Q|^2 \int_0^{\ell(Q)} t dt dx \leq C\epsilon_0 |Q|,$$

where in the last step we have used (6.16)(ii).

It is also easy to handle $R_t^{(1)} \nabla_{\|} F_Q$. Indeed $E_t = E_t^2$, so that

$$R_t^{(1)} = (T_t A_{\|}) E_t (E_t - P_t)$$

By the definition of T_t, Lemma 3.3 and Lemma 3.11, we have that

$$\|(T_t A_{\|}) E_t\|_{2\to 2} \leq C\epsilon_0. \tag{6.17}$$

Thus,

$$\int_0^{\ell(Q)} \int_Q |R_t^{(1)} \nabla_{\|} F_Q|^2 dx \frac{dt}{t} \leq C\epsilon_0 \iint_{\mathbb{R}^{n+1}_+} |(E_t - P_t) \nabla_{\|} F_Q|^2 \frac{dxdt}{t} \leq C\epsilon_0 |Q|,$$

where in the last step we have used (6.16)(i), as well as the boundedness on L^2 of

$$g \to \left(\int_0^\infty |(E_t - P_t)g|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

It remains to treat the contribution of the term $R_t^{(2)} \nabla_\| F_Q$. By (6.16)(i), it will be enough to establish the square function bound

$$|||R_t^{(2)} \nabla_\| F_Q||| \le C\epsilon_0 \|\nabla_\| F_Q\|_2.$$

To this end, we write

$$R_t^{(2)} \nabla_\| F_Q = R_t^{(2)}(I - P_t)\nabla_\| F_Q \; + \; R_t^{(2)} P_t \nabla_\| F_Q, \tag{6.18}$$

where I denotes the identity operator. The last term is easy to handle. We note that $R_t^{(2)} 1 = 0$, and therefore by Lemmas 2.7, 2.8, 3.3 and 3.11, the operator $R_t^{(2)}$ satisfies the hypotheses of Lemma 3.5 with bound on the order of $C\epsilon_0$. Thus,

$$|||R_t^{(2)} P_t \nabla_\| F_Q||| \le C\epsilon_0 \, |||t\nabla_\| P_t \nabla_\| F_Q||| \le C\epsilon_0 \, \|\nabla_\| F_Q\|_2,$$

where the last inequality is standard Littlewood-Paley theory.

By the definition of $R_t^{(2)}$, we may further decompose the first summand on the right side of (6.18) as

$$(T_t A_\|) E_t Q_t \nabla_\| F_Q \; - \; T_t A_\| \nabla_\| (I - P_t) F_Q \equiv \mathbf{I} - \mathbf{II},$$

where $Q_t \equiv P_t(I - P_t)$ satisfies $|||Q_t|||_{op} \le C$. Then by (6.17), we have

$$|||\mathbf{I}||| \le C\epsilon_0 \|\nabla_\| F_Q\|_2.$$

Next, by definition of T_t, we see that

$$\begin{aligned} \mathbf{II} = \; & \theta_t \tilde{\epsilon} \, \nabla_\| \left(\left(I + t^2 (L_1)_\| \right)^{-1} - I \right) (I - P_t) F_Q \; = \; -\theta_t \tilde{\epsilon} \, \nabla_\| F_Q \\ & + \; \theta_t \tilde{\epsilon} \, \nabla_\| P_t F_Q \; + \; \theta_t \tilde{\epsilon} \, \nabla_\| \left(I + t^2 (L_1)_\| \right)^{-1} (I - P_t) F_Q \equiv \; \mathbf{II}_1 + \mathbf{II}_2 + \mathbf{II}_3. \end{aligned}$$

By Lemma 5.17,

$$|||\mathbf{II}_1||| \le C\epsilon_0 \|\nabla_\| F_Q\|.$$

Moreover, by Lemma 2.8 and the fact that $\|t\nabla_\| (1 + t^2 (L_1)_\|)^{-1}\|_{2\to 2} \le C$, we obtain that

$$|||\mathbf{II}_3||| \le C\epsilon_0 |||t^{-1} I_1 (I - P_t) \sqrt{-\Delta} F_Q||| \le C\epsilon_0 \|\nabla_\| F_Q\|_2,$$

where $I_1 = (-\Delta)^{-1/2}$ is the fractional integral operator of order one on $\mathbb{R}^n$, and where we have used the Littlewood-Paley inequality

$$|||t^{-1} I_1 (I - P_t)|||_{op} \le C.$$

The latter estimate holds by Plancherel's Theorem, since

$$\left| \frac{1}{t|\xi|} (1 - \hat{\phi}(t\xi)) \right| \; \le \; C \min\left(t|\xi|, \frac{1}{t|\xi|} \right), \tag{6.19}$$

if $\phi_t(x) = t^{-n} \phi_t(x/t))$, the convolution kernel of P_t, is chosen so that $\int_{\mathbb{R}^n} x\phi_t(x) dx = 0$.

Finally, it remains only to consider the term $\mathbf{II}_2$. Now

$$\mathbf{II}_2 = \theta_t \tilde{\epsilon} P_t \nabla_\| F_Q,$$

so we need that $|||\theta_t \tilde{\epsilon} P_t|||_{op} \le C\epsilon_0$. By Lemmas 5.17, 3.2 and 2.7, $|\theta_t \tilde{\epsilon}|^2 t^{-1} dx dt$ is a Carleson measure with norm at most $C\epsilon_0^2$, so it is enough to bound $|||\theta_t \tilde{\epsilon} P_t - (\theta_t \tilde{\epsilon}) P_t|||_{op}$. We may choose P_t to be of the form $P_t = \tilde{P}_t^2$, where $\tilde{P}_t$ is of the same type. Set

$$R_t \equiv \theta_t \tilde{\epsilon} \tilde{P}_t - (\theta_t \tilde{\epsilon}) \tilde{P}_t,$$

which satisfies the hypothesis of Lemma 3.5 with bound $C\epsilon_0$. Thus,

$$|||\theta_t\tilde{\epsilon}P_t - (\theta_t\tilde{\epsilon})P_t|||_{op} \equiv |||R_t\tilde{P}_t|||_{op} \le C\epsilon_0|||t\nabla\tilde{P}_t|||_{op} \le C\epsilon_0.$$

This concludes the proof of Lemma 6.9, and hence that of the square function bound (5.7). $\square$

7. P T 1.11: (5.8)

We shall consider separately the cases $t > 0$ and $t < 0$, and since the proof is the same in each case we treat only the former. More precisely, we shall prove

$$\sup_{0<\eta<10^{-10}} \sup_{t>0} \|\partial_t S_t^{1,\eta} f\|_2 \le C\|f\|_2 + C\epsilon_0 \left(\mathbf{M}^+ + \mathbf{M}^-\right), \tag{7.1}$$

where $\mathbf{M}^\pm$ are defined in (6.2). We begin by reducing matters to the case $t \ge 4\eta$. Suppose that $0 \le t < 4\eta$. We claim that

$$|\partial_t S_t^{1,\eta} f(x) - D_{n+1}S_{4\eta}^{1,\eta} f(x)| \le CMf(x).$$

Indeed, let $K_t^\eta(x,y)$ denote the kernel of $\partial_t S_t^{1,\eta}$, i.e.,

$$K_t^\eta(x,y) \equiv \partial_t\left(\varphi_\eta * \Gamma_1(x,\cdot,y,0)\right)(t).$$

To prove the claim, it is enough to establish the following estimate:

$$|K_t^\eta(x,y) - K_{4\eta}^\eta(x,y)| \le C\left(\frac{1_{\{|x-y|\le 10\eta\}}}{\eta|x-y|^{n-1}} + \frac{\eta}{|x-y|^{n+1}}1_{\{|x-y|>10\eta\}}\right).$$

In turn, the case $|x-y| \le 10\eta$ of the latter bound follows directly from (4.10). On the other hand, if $|x-y| > 10\eta$, we have by Lemma 2.2 that

$$\begin{aligned}|K_t^\eta(x,y) - K_{4\eta}^\eta(x,y)| &= \left|\int \varphi_\eta(s)\left(D_{n+1}\Gamma(x,t-s,y,0) - D_{n+1}\Gamma(x,4\eta-s,y,0)\right)ds\right| \\ &\le C\int|\varphi_\eta(s)|\frac{|4\eta-t|}{|x-y|^{n+1}}ds \le C\frac{\eta}{|x-y|^{n+1}}.\end{aligned}$$

Having proved the claim, we fix $t_0 \ge 4\eta$, and use (1.3) to obtain, for each $y \in \mathbb{R}^n$,

$$\begin{aligned}|(D_{n+1}S_{t_0}^{1,\eta})f(y)| &\le C\left(\fint\!\!\!\fint_{B((y,t_0),t_0/2)} |\partial_\tau S_\tau^{1,\eta} f(x)|^2 dx d\tau\right)^{\frac{1}{2}} \\ &\le C\left(\fint\!\!\!\fint_{B((y,t_0),t_0/2)} |\partial_\tau S_\tau^{1,\eta} f - \partial_\tau S_\tau^{0,\eta} f|^2 dx d\tau\right)^{\frac{1}{2}} + \text{“OK”},\end{aligned}$$

where $\|\text{“OK”}\|_{L^2(\mathbb{R}^n)} \le C\|f\|_2$ uniformly in t_0, by our hypotheses regarding L_0, and where we have used that $u_\eta(x,t) \equiv S_t^{1,\eta} f(x)$ solves $L_1 u_\eta = 0$ in $\{t > \eta\}$. Consequently,

$$\|(D_{n+1}S_{t_0}^{1,\eta})f\|_2^2 \le C\|f\|_2 + C\frac{1}{t_0}\int_{t_0/2}^{3t_0/2}\int_{\mathbb{R}^n} |\partial_\tau S_\tau^{1,\eta} f - \partial_\tau S_\tau^{0,\eta} f|^2 dx d\tau.$$

As in the section 6,

$$\partial_\tau S_\tau^{1,\eta} f(x) - \partial_\tau S_\tau^{0,\eta} f(x) = \partial_\tau\left(L_0^{-1}\operatorname{div}\epsilon\nabla S_{(\cdot)}^{1,\eta} f\right)(x,\tau).$$

Thus, it is enough to prove that for every $\Psi \in C_0^\infty\left(\mathbb{R}^n \times (\frac{t_0}{2}, \frac{3t_0}{2})\right)$, with $t_0^{-1/2}\|\Psi\|_2 = 1$, and for each $\eta > 0$ and $\delta > 0$ sufficiently small, we have

$$\left|\frac{1}{t_0}\iint_{\mathbb{R}^{n+1}} \epsilon(y)\nabla S_s^{1,\eta} f(y) \cdot \overline{\nabla(L_0^*)^{-1}\left(D_{n+1}\Psi_\delta\right)(y,s)}\,dyds\right| \leq C\epsilon_0(\mathbf{M}^+ + \mathbf{M}^-), \tag{7.2}$$

where again $\Psi_\delta \equiv \varphi_\delta * \Psi$. We may then obtain (7.1) by taking first a limit as $\delta \to 0$, and then a supremum over all such Ψ.

To prove (7.2), we begin by splitting the integral on the left hand side into

$$\frac{1}{t_0}\left\{\int_{-t_0/4}^{t_0/4}\int_{\mathbb{R}^n} + \int_{t_0/4}^{4t_0}\int_{\mathbb{R}^n} + \int_{4t_0}^{\infty}\int_{\mathbb{R}^n} + \int_{-\infty}^{-t_0/4}\int_{\mathbb{R}^n}\right\} \equiv I + II + III + IV. \tag{7.3}$$

Since $\nabla(L_0^*)^{-1}$ div is bounded on $L^2(\mathbb{R}^{n+1})$, by Cauchy-Schwarz and our assumptions on Ψ, we have that

$$|II| \leq C\epsilon_0\left(t_0^{-1}\int_{t_0/4}^{4t_0}\int_{\mathbb{R}^n}|\nabla S_s^{1,\eta} f(y)|^2 dyds\right)^{1/2} \leq C\epsilon_0 \sup_{t>0}\|\nabla S_t^{1,\eta} f\|_2.$$

Next we consider terms III and IV. These may be handled in the same way, so we treat only III explicitly. We use (6.3) to write

$$\nabla(L_0^*)^{-1}(D_{n+1}\Psi_\delta)(y,s) = \int \nabla_{y,s}\partial_s S_{s-\tau}^{L_0^*,\delta}\left(\Psi(\cdot,\tau)\right)(y)\,d\tau, \tag{7.4}$$

so that

$$\begin{aligned} III &= t_0^{-1}\int\int_{4t_0}^{\infty}\int_{\mathbb{R}^n}\left(\partial_\tau S_{\tau-s}^{0}\nabla\right)\cdot\epsilon\nabla S_s^{1,\eta} f(x)\Psi_\delta(x,\tau)dxdsd\tau \\ &= \frac{1}{t_0}\int\int_{2\tau}^{\infty}\int_{\mathbb{R}^n} - \frac{1}{t_0}\int\int_{2\tau}^{4t_0}\int_{\mathbb{R}^n} \equiv \widetilde{III} - \text{ error.} \end{aligned}$$

In the error term, $s - \tau \approx s \approx \tau \approx t_0$, if δ is sufficiently small, given the support constraints on Ψ. Thus by Cauchy-Schwarz and Lemma 2.8 (i), the absolute value of the error term is bounded by $C\epsilon_0 \sup_{t>0}\|\nabla S_t^{1,\eta} f\|_2$. The remaining term is

$$\begin{aligned} \widetilde{III} &= t_0^{-1}\int\int_{2\tau}^{\infty}\int_{\mathbb{R}^n}\left(\partial_\tau S_{\tau-s}^{0}\nabla\right)\cdot\epsilon\nabla S_s^{1,\eta} f(x)\Psi_\delta(x,\tau)dxdsd\tau \\ &= t_0^{-1}\int\lim_{R\to\infty}\int_{\mathbb{R}^n}\left\{\int_{2\tau}^{2R}\left(\partial_\tau S_{\tau-s}^{0}\nabla\right)\cdot\epsilon\nabla S_s^{1,\eta} f(x)ds\right\}\Psi_\delta(x,\tau)dxd\tau \\ &\equiv t_0^{-1}\int\lim_{R\to\infty}\int_{\mathbb{R}^n} H_R(x,\tau)\,\Psi_\delta(x,\tau)dxd\tau, \end{aligned}$$

where the expression in curly brackets equals

$$\begin{aligned} H_R(x,\tau) &= -\int_\tau^R \partial_t\left(\int_{2t}^{2R}\left(\partial_t S_{t-s}^{0}\nabla\right)\cdot\epsilon\nabla S_s^{1,\eta} f(x)ds\right)dt \\ &= -\int_\tau^R \partial_t\left(\int_t^{2R-t}\left(D_{n+1}S_{-s}^{0}\nabla\right)\cdot\epsilon\nabla S_{t+s}^{1,\eta} f(x)ds\right)dt \\ &= \int_\tau^R\left(D_{n+1}S_{-t}^{0}\nabla\right)\cdot\epsilon\nabla S_{2t}^{1,\eta} f(x)dt - \int_\tau^R\left(D_{n+1}S_{t-2R}^{0}\nabla\right)\cdot\epsilon\nabla S_{2R}^{1,\eta} f(x)dt \\ &\quad - \int_\tau^R\left(\int_t^{2R-t}\left(D_{n+1}S_{-s}^{0}\nabla\right)\cdot\epsilon\nabla\partial_t S_{t+s}^{1,\eta} f(x)ds\right)dt \\ &\equiv H_R'(x,\tau) - H_R''(x,\tau) - H_R'''(x,\tau). \end{aligned}$$

Since $|t-2R| \approx R$, we have that by Lemma 2.8 (i),

$$\sup_{\tau,R:0<\tau<R} \|H_R''(\cdot,\tau)\|_2 \leq C\epsilon_0 \sup_{t>0} \|\nabla S_t^{1,\eta} f\|_2,$$

from which the desired bound for the corresponding part of $\widetilde{III}$ follows readily. Similarly, we may treat the contribution of $H_R'(x,\tau)$ by a direct application of the following Lemma, which is really the deep result in this section.

Lemma 7.5. *Let a, b denote non-zero real constants. We then have that*

$$\sup_{0\leq\tau_1<\tau_2<\infty} \left\| \int_{\tau_1}^{\tau_2} \left(D_{n+1} S_{at}^{0} \nabla\right) \cdot \epsilon \nabla S_{bt}^{1,\eta} f dt \right\|_2 \leq C(a,b)\epsilon_0 \left(\mathbf{M}^+ + \mathbf{M}^-\right).$$

We defer for the moment the proof of this Lemma, and consider now

$$H_R'''(x,\tau) = \int_\tau^R \int_{2t}^{2R} \left(\partial_t S_{t-s}^{0} \nabla\right) \cdot \epsilon \partial_s \nabla S_s^{1,\eta} f(x)\, ds dt.$$

Then for $h \in L^2(\mathbb{R}^n)$, with $\|h\|_2 = 1$, we have

$$\left|\langle h, H_R'''(\cdot,\tau)\rangle\right| = \left| \int_\tau^R \int_{2t}^{2R} \langle \nabla D_{n+1} S_{s-t}^{L_0^*} h, \epsilon\, \partial_s \nabla S_s^{1,\eta} f\rangle\, ds dt \right|, \tag{7.6}$$

where we have used that $adj(S_{t-s}^{0}) = S_{s-t}^{L_0^*}$ (recall that adj indicates that we have taken the adjoint in the x, y variables only, whereas $S_t^{L_0^*}$ is the single layer potential operator associated to L_0^*). Thus, (7.6) is dominated by

$$C\epsilon_0 \left(\int_0^\infty \int_{2t}^\infty \|\nabla \partial_s S_{s-t}^{L_0^*} h\|_2^2\, ds dt \right)^{\frac{1}{2}} \left(\int_0^\infty \|\partial_s \nabla S_s^{1,\eta} f\|_2^2 \int_0^{s/2} dt ds \right)^{\frac{1}{2}} \equiv C\epsilon_0\, B_1 \cdot B_2.$$

Note that $B_2 = C|||s\nabla\partial_s S_s^{1,\eta} f|||$. Similarly, the change of variable $s \to s+t$ yields that $B_1 = |||s\partial_s \nabla S_s^{L_0^*} h||| \leq C\|h\|_2 = C$. A suitable bound follows for the contribution of H_R'''.

It remains to consider the term I in (7.3), which we shall also treat via Lemma 7.5. Again using (7.4), and that for small δ, Ψ_δ is supported in $\{t_0/2 < \tau < 3t_0/2\}$, we write

$$\begin{aligned} I &= t_0^{-1} \int \int_{-t_0/4}^{t_0/4} \int_{\mathbb{R}^n} \left(\partial_\tau S_{\tau-s}^{0} \nabla\right) \cdot \epsilon \nabla S_s^{1,\eta} f(x) \Psi_\delta(x,\tau) dx ds d\tau \\ &= \frac{1}{t_0} \int \int_{-\tau/2}^{\tau/2} \int_{\mathbb{R}^n} - \frac{1}{t_0} \int \int_{t_0/4<|s|<\tau/2} \int_{\mathbb{R}^n} \equiv \widetilde{I} - \text{ error}. \end{aligned}$$

By Cauchy-Schwarz and Lemma 2.8 (i), the absolute value of the error term is bounded by $C\epsilon_0 \sup_{t>0} \|\nabla S_t^{1,\eta} f\|_2$, since $\tau - s \approx \tau \approx t_0$. The remaining term splits into

$$\begin{aligned} \widetilde{I}_+ &= t_0^{-1} \int \int_{\mathbb{R}^n} \left\{ \int_0^{\tau/2} \left(\partial_\tau S_{\tau-s}^{0} \nabla\right) \cdot \epsilon \nabla S_s^{1,\eta} f(x) ds \right\} \Psi_\delta(x,\tau) dx d\tau \\ &\equiv t_0^{-1} \int \int_{\mathbb{R}^n} F(x,\tau)\, \Psi_\delta(x,\tau) dx d\tau, \end{aligned}$$

plus a similar term $\widetilde{I}_-$, which may be treated by the same arguments, in which the expression in curly brackets has domain of integration $(-\tau/2, 0)$. Now,

$$\begin{aligned}F(\cdot,\tau) = \int_0^\tau \partial_t \left(\int_0^{t/2} \left(\partial_t S^0_{t-s}\nabla\right)\cdot \epsilon\nabla S^{1,\eta}_s f ds\right) dt \\ = \int_0^\tau \partial_t \left(\int_{t/2}^t \left(D_{n+1} S^0_s \nabla\right)\cdot \epsilon\nabla S^{1,\eta}_{t-s} f ds\right) dt \\ = \int_0^\tau \left(\partial_t S^0_t \nabla\right)\cdot \epsilon\nabla S^{1,\eta}_0 f dt - \int_0^\tau \left(D_{n+1} S^0_{t/2}\nabla\right)\cdot \epsilon\nabla S^{1,\eta}_{t/2} f dt \\ + \int_0^\tau \int_0^{t/2} \left(\partial_t S^0_{t-s}\nabla\right)\cdot \epsilon\nabla \partial_s S^{1,\eta}_s f ds dt \;\equiv\; F' - F'' + F'''.\end{aligned}$$

We may estimate the contribution of F'' directly via Lemma 7.5. Also,

$$F'(\cdot,\tau) = \left(S^0_\tau\nabla\right)\cdot\epsilon\nabla S^{1,\eta}_0 f - \left(S^0_{0^+}\nabla\right)\cdot \epsilon\nabla S^{1,\eta}_0 f,$$

so by our hypotheses concerning L_0,

$$\sup_\tau \|F'(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \le C\epsilon_0 \sup_{t>0} \|\nabla S^{1,\eta}_t f\|_2.$$

We therefore obtain a permissible bound for the contribution of F'. We also have that

$$\begin{aligned}F'''(\cdot,\tau) = \int_0^\tau \int_0^{t/2} \partial_t \left(\left(S^0_{t-s} - S^0_t\right)\nabla\right)\cdot \epsilon\nabla\partial_s S^{1,\eta}_s f ds dt \\ + \int_0^\tau \left(\partial_t S^0_t\nabla\right)\cdot\epsilon\nabla S^{1,\eta}_{t/2} f dt - \int_0^\tau \left(\partial_t S^0_t \nabla\right)\cdot \epsilon\nabla S^{1,\eta}_0 f dt.\end{aligned} \tag{7.7}$$

In turn, the last term equals $-F'$, and the middle summand may be handled via Lemma 7.5. The first summand on the right hand side of (7.7) equals

$$-\int_0^\tau \int_0^{t/2} \int_0^s \partial_t^2 (S^0_{t-\sigma}\nabla)\cdot \epsilon\nabla\partial_s S^{1,\eta}_s f(x) d\sigma ds dt.$$

Dualizing against $h \in L^2(\mathbb{R}^n)$, with $\|h\|_2 = 1$, we see that it is enough to consider

$$\begin{aligned}&\left|\int_0^\tau \int_0^\infty \int_0^\infty 1_{\{\sigma<s<t/2\}} \langle \nabla D^2_{n+1} S^{L_0^*}_{\sigma-t} h, \epsilon D_{n+1}\nabla S^{1,\eta}_s f\rangle d\sigma ds dt\right| \\ &\le C\epsilon_0 \left(\int_0^\infty \int_0^\infty \int_0^\infty 1_{\{\sigma<s<t/2\}} s^{-\frac12} t^{\frac32} \|\nabla D^2_{n+1} S^{L_0^*}_{\sigma-t} h\|_2^2 \, d\sigma ds dt\right)^{\frac12} \\ &\quad\times \left(\int_0^\infty \int_0^\infty \int_0^\infty 1_{\{\sigma<s<t/2\}} s^{\frac12} t^{-\frac32} \|\partial_s \nabla S^{1,\eta}_s f\|_2^2 d\sigma ds dt\right)^{\frac12} \\ &\equiv C\epsilon_0 B_3\cdot B_4.\end{aligned}$$

Now,

$$B_4 = \left(\int_0^\infty \|\partial_s\nabla S^{1,\eta}_s f\|_2^2 \left(\int_0^s d\sigma \int_{2s}^\infty \frac{s^{1/2}}{t^{3/2}} dt\right) ds\right)^{\frac12} = C|||s\nabla\partial_s S^{1,\eta}_s f|||.$$

Similarly, the change of variable $t \to t+\sigma$ yields the bound

$$B_3 = \left(\int_0^\infty \int_0^\infty \int_0^\infty 1_{\{\sigma<s<(t+\sigma)/2\}} s^{-\frac{1}{2}} (t+\sigma)^{\frac{3}{2}} \|\nabla D_{n+1}^2 S_{-t}^{L_0^*} h\|_2^2 d\sigma ds dt\right)^{\frac{1}{2}}$$
$$\leq C\left(\int_0^\infty t^{\frac{3}{2}} \|\nabla \partial_t^2 S_{-t}^{L_0^*} h\|_2^2 \int_0^t s^{-\frac{1}{2}} \int_0^s d\sigma ds dt\right)^{\frac{1}{2}} = C|||t^2 \nabla \partial_t^2 S_{-t}^{L_0^*} h||| \leq C\|h\|_2 = C,$$

and the desired estimate for the contribution of F''' now follows.

To complete the proof of estimate (5.8), it therefore remains to prove Lemma 7.5.

Proof of Lemma 7.5. For the sake of simplicity of notation, we shall treat the case $a = 2$, $b = 1$, as the general case follows via the same argument.

As above we dualize against $h \in L^2(\mathbb{R}^n)$, so that it is enough to consider

$$\int_{\tau_1}^{\tau_2} \langle \nabla \partial_t S_{-2t}^{L_0^*} h, \epsilon \nabla S_t^{1,\eta} f\rangle dt = -\int_{\tau_1}^{\tau_2} \langle \nabla \partial_t^2 S_{-2t}^{L_0^*} h, \epsilon \nabla S_t^{1,\eta} f\rangle t dt$$
$$- \int_{\tau_1}^{\tau_2} \langle \nabla \partial_t S_{-2t}^{L_0^*} h, \epsilon \nabla \partial_t S_t^{1,\eta} f\rangle t dt \; + \; \text{boundary}, \tag{7.8}$$

where we have integrated by parts in t, and where the boundary term is dominated by

$$C\epsilon_0 \left(\sup_{\tau>0} \|\tau \nabla \partial_\tau S_{-2\tau}^{L_0^*} h\|_2\right)\left(\sup_{\tau>0} \|\nabla S_\tau^{1,\eta} f\|_2\right) \; \leq \; C\epsilon_0 \sup_{\tau>0} \|\nabla S_\tau^{1,\eta} f\|_2,$$

as desired. Here, the last inequality follows from Lemma 2.8 (*ii*). Moreover, by Cauchy-Schwarz, the middle term on the right hand side of (7.8) is no larger than

$$C\epsilon_0\, |||t \nabla \partial_t S_{-2t}^{L_0^*} h||| \cdot |||t \nabla \partial_t S_t^{1,\eta} f||| \; \leq \; C\epsilon_0 |||t \nabla \partial_t S_t^{1,\eta} f|||.$$

In the first term on the right hand side of (7.8), we integrate by parts again in t, to obtain

$$\frac{1}{2} \int_{\tau_1}^{\tau_2} \langle \nabla \partial_t^3 S_{-2t}^{L_0^*} h, \epsilon \nabla S_t^{1,\eta} f\rangle t^2 dt \; + \; \text{Errors}, \tag{7.9}$$

where the error terms correspond to the last two terms in (7.8) and are handled in a similar fashion. Turning to the main term in (7.9), we note that

$$\frac{1}{2} \partial_t^3 S_{-2t}^{L_0^*} h = \partial_s \partial_t^2 S_{-t-s}^{L_0^*} h|_{s=t}.$$

Now set $g \equiv \partial_t^2 S_{-t}^{L_0^*} h$. Let u solve

$$\begin{cases} L_0^* u = 0 & \text{in } \mathbb{R}_-^{n+1} \\ u(\cdot, 0) = g \end{cases}.$$

By invertibility of the layer potentials for L_0^*, and by uniqueness, we have that

$$u(\cdot, -s) = \mathcal{D}_{-s}^{L_0^*} \left(\frac{1}{2} I + K^{L_0^*}\right)^{-1} g.$$

On the other hand, we also have that $u(\cdot, -s) = \partial_t^2 S_{-t-s}^{L_0^*} h$. Consequently,

$$\partial_s \nabla u(\cdot, -s) = \partial_s \nabla \mathcal{D}_{-s}^{L_0^*} \left(\frac{1}{2} I + K^{L_0^*}\right)^{-1} g = \partial_s \nabla \partial_t^2 S_{-t-s}^{L_0^*} h.$$

Setting $s = t$, we have that

$$\frac{1}{2}\nabla\partial_t^3 S_{-2t}^{L_0^*} h = -D_{n+1}\nabla\mathcal{D}_{-t}^{L_0^*}\left(\frac{1}{2}I + K^{L_0^*}\right)^{-1} g = -D_{n+1}\nabla\mathcal{D}_{-t}^{L_0^*}\left(\frac{1}{2}I + K^{L_0^*}\right)^{-1}\partial_t^2 S_{-t}^{L_0^*} h.$$

But, $\mathcal{D}_{-t}^{L_0^*} = (S_{-t}^{L_0^*}\overline{\partial_{\nu_0}})$, where $\overline{\partial_{\nu_0}}$ denotes conjugate exterior co-normal differentiation for L_0. Thus,

$$adj\left(\nabla D_{n+1}\mathcal{D}_{-t}^{L_0^*}\right) = \left(\partial_{\nu_0}\partial_t S_t^0\nabla\right).$$

Therefore, the main term in (7.9) equals in absolute value

$$\left|\int_{\tau_1}^{\tau_2}\left\langle\left(\frac{1}{2}I + K^{L_0^*}\right)^{-1}\partial_t^2 S_{-t}^{L_0^*} h, \left(\partial_{\nu_0} D_{n+1} S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f\right\rangle t^2 dt\right|$$
$$\leq C|||t\partial_t^2 S_{-t}^{L_0^*} h|||\cdot|||t^2\left(\nabla D_{n+1} S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f||| \leq C|||t^2\left(\nabla\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f|||.$$

To conclude the proof of Lemma 7.5, it then suffices to prove that

$$|||t^2\left(\nabla\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_t^{1,\eta} f||| \leq C\epsilon_0\,\mathbf{M}^+. \tag{7.10}$$

To this end, we first prove a lemma that will allow us to reduce matters to (6.10).

Lemma 7.11. *For $k \in \mathbb{Z}$, set $t_k \equiv 2^{k-1}$. Then*

$$\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^{k+2}}\int_{\mathbb{R}^n}|\nabla S_t^{1,\eta} f(x) - \nabla S_{t_k}^{1,\eta} f(x)|^2\frac{dxdt}{t} \leq C|||t\nabla\partial_t S_t^{1,\eta} f|||_2. \tag{7.12}$$

Let us momentarily take the lemma for granted, and deduce (7.10). Combining Lemma 2.8 (*i*), Lemma 2.11 and Lemma 7.11, we may replace the square of the left hand side of (7.10) by

$$\sum_{k=-\infty}^{\infty}\int_{2^k}^{2^{k+1}}\int_{\mathbb{R}^n}|t^2\left(\nabla\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_{t_k}^{1,\eta} f(x)|^2\frac{dxdt}{t}.$$

Since $u_k(\cdot,t) \equiv \left(\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_{t_k}^{1,\eta} f$ solves $L_0 u_k = 0$ in the upper half space, we may use Caccioppoli's inequality in Whitney boxes to reduce matters to considering

$$\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^{k+2}}\int_{\mathbb{R}^n}|t\left(\partial_t S_t^0\nabla\right)\cdot\epsilon\nabla S_{t_k}^{1,\eta} f(x)|^2\frac{dxdt}{t}.$$

Applying Lemma 2.8 (*i*) and Lemma 7.11 again, along with (6.10), we obtain (7.10).

Proof of Lemma 7.11. The left hand side of (7.12) equals

$$\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^{k+2}}\int_{\mathbb{R}^n}\left|\frac{1}{\sqrt{t}}\int_{t_k}^{t}\nabla\partial_s S_s^{1,\eta} f(x)ds\right|^2 dxdt$$
$$\leq C\sum_{k=-\infty}^{\infty}\iint_{\mathbb{R}^{n+1}}\left|\fint_{t_k}^{t} 1_{\{2^{k-1}\leq s<2^{k+2}\}}\sqrt{s}\nabla\partial_s S_s^{1,\eta} f(x)ds\right|^2 dtdx.$$

The desired bound now follows from the Hardy-Littlewood maximal theorem. $\square$

This concludes the proof Lemma 7.5, and thus also that of Theorem 1.11 $\square$

8. P T 1.12:

Let $L \equiv -\operatorname{div} A\nabla$, where A is real, symmetric, L^∞, t-independent and uniformly elliptic. In this section, we show that the layer potentials associated to L are bounded; we defer the proof of invertibility to the next section. By the classical de Giorgi-Nash Theorem, estimates (1.2) and (1.3) hold for solutions of $Lu = 0$. By Lemma 5.2 and Lemma 4.8, in order to establish boundedness of the layer potentials, it suffices to prove

$$\sup_{t\neq 0} \|\partial_t \mathcal{S}_t f\|_2 \leq C\|f\|_2 \tag{8.1}$$

and

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |t\partial_t^2 \mathcal{S}_t f|^2 dx \frac{dt}{|t|} \leq C\|f\|_2. \tag{8.2}$$

By Lemma 2.2, the kernel $K_t(x,y) \equiv \partial_t\Gamma(x,t,y,0)$ satisfies the standard Calderón-Zygmund estimates

$$|K_t(x,y)| \leq \frac{c}{|x-y|^n} \tag{8.3a}$$

$$|K_t(x,y+h) - K_t(x,y)| + |K_t(x+h,y) - K_t(x,y)| \leq C\frac{|h|^\alpha}{|x-y|^{n+\alpha}}, \tag{8.3b}$$

uniformly in t, where the later inequality holds for some $\alpha > 0$ whenever $|x-y| > 2|h|$. In addition, the kernel

$$\psi_t(x,y) \equiv t\partial_t^2\Gamma(x,t,y,0)$$

satisfies the standard Littlewood-Paley kernel conditions

$$\begin{aligned} |\psi_t(x,y)| &\leq \frac{|t|}{(|t|+|x-y|)^{n+1}} \\ |\psi_t(x,y+h) - \psi_t(x,y)| &\leq \frac{C|t|\,|h|^\alpha}{(|t|+|x-y|)^{n+1+\alpha}} \leq \frac{C|h|^\alpha}{(|t|+|x-y|)^{n+\alpha}} \end{aligned} \tag{8.4}$$

for some $\alpha > 0$, whenever $|h| \leq \frac{1}{2}|x-y|$ or $|h| \leq |t|/2$.

The bound (8.2) will be deduced from the following "local" Tb Theorem for square functions

Theorem 8.5. *Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x,y) f(y)dy$, where $\psi_t(x,y)$ satisfies* (8.4)*. Suppose also that there exists a system $\{b_Q\}$ of functions indexed by cubes $Q \subseteq \mathbb{R}^n$ such that for each cube Q*

(i) $\int_{\mathbb{R}^n} |b_Q|^2 \leq C|Q|$

(ii) $\int_0^{\ell(Q)}\int_Q |\theta_t b_Q(x)|^2 \frac{dxdt}{t} \leq C|Q|$

(iii) $\frac{1}{C}|Q| \leq \Re e \int_Q b_Q$.

Then we have the square function bound

$$|||\theta_t f||| \leq C\|f\|_2.$$

We omit the proof here. A direct proof of the present formulation of Theorem 8.5 may be found in [A2] or [H2], although we note that the theorem and its proof were already implicit in the proof of the Kato square root conjecture [HMc], [HLMc] and [AHLMcT]; see also the works [Ch], [S] and [AT] for some important antecedents.

We shall deduce estimate (8.1) as a consequence of the following extension of a local Tb Theorem for singular integrals introduced by M. Christ [Ch] in connection with the theory of analytic capacity. A 1-dimensional version of the present result, valid for "perfect

dyadic" Calderón-Zygmund kernels, appears in [AHMTT]. A self-contained proof of the more general formulation below may be found in [H3]. Alternatively, the result of [AY] may be combined with that of [AHMTT] to deduce the general case (in the slightly sharper form $q = 2$). In the sequel, we let T^{tr} denote the transpose of the operator T.

Theorem 8.6. *Let T be a singular integral operator associated to a kernel K satisfying* (8.3)*, and suppose that K satisfies the generalized truncation condition $K(x, y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose also that there exist pseudo-accretive systems $\{b^1_Q\}$, $\{b^2_Q\}$ such that b^1_Q and b^2_Q are supported in Q, and*

(i) $\int_Q \left(|b^1_Q|^q + |b^2_Q|^q\right) \le C|Q|$*, for some* $q > 2$

(ii) $\int_Q \left(|Tb^1_Q|^2 + |T^{\mathrm{tr}} b^2_Q|^2\right) \le C|Q|$

(iii) $\frac{1}{C}|Q| \le \min\left(\Re e \int_Q b^1_Q, \Re e \int_Q b^2_Q\right)$.

Then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, with bound independent of $\|K\|_\infty$.

Let us first show that Theorem 8.5 implies (8.2). As usual, we may restrict our attention to the case $t > 0$. As above let $\psi_t(x, y) \equiv t\partial_t^2\Gamma(x, t, y, 0)$, so that

$$t\partial_t^2 S_t f(x) \equiv \theta_t f(x) = \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy.$$

By Theorem 8.5, it suffices to construct a system $\{b_Q\}$ satisfying the hypotheses (i), (ii) and (iii) of the Theorem.

Our functions b_Q will be normalized Poisson kernels. Given a cube $Q \subset \mathbb{R}^n$, let x_Q denote its center, and let $\ell(Q)$ denote its side length. We define

$$A^+_Q \equiv (x_Q, \ell(Q)) \in \mathbb{R}^{n+1}_+, \quad A^-_Q \equiv (x_Q, -\ell(Q)) \in \mathbb{R}^{n+1}_-.$$

Given $X^+ \in \mathbb{R}^{n+1}_+$, $X^- \in \mathbb{R}^{n+1}_-$, let $k^{X^+}_+(y)$, $k^{X^-}_-(y)$ denote, respectively, the Poisson kernels for L in the upper and lower half spaces, and let $G^+(X, Y)$, $G^-(X, Y)$ denote the corresponding Green functions, so that

$$k^{X^+}_+(y) \equiv \frac{\partial G^+}{\partial \nu^+_y}(X^+, y, 0), \quad k^{X^-}_-(y) \equiv \frac{\partial G^-}{\partial \nu^-_y}(X^-, y, 0),$$

where $\frac{\partial}{\partial \nu^+_y}$, $\frac{\partial}{\partial \nu^-_y}$ denote the co-normal derivatives at the point $y \in \partial\mathbb{R}^{n+1}_+$, $\partial\mathbb{R}^{n+1}_-$ respectively. We now set

$$b_Q \equiv |Q|\, k^{A^-_Q}_-. \tag{8.7}$$

We recall the following fundamental result of Jerison and Kenig [JK1] (see also [K, pp 63-64]), which amounts to the solvability of (D2) in the lower half-space:

Theorem 8.8. [JK1] *Suppose that $L = -\operatorname{div} A\nabla$, where A is real, symmetric, $(n+1)\times(n+1)$, t-independent, L^∞ and uniformly elliptic. Then there exists $\varepsilon_1 \equiv \varepsilon_1(n, \lambda, \Lambda)$ such that for all $0 \le \varepsilon < \varepsilon_1$ and for every cube Q,*

$$\int_{\mathbb{R}^n} (k^{A^-_Q}_-(y))^{2+\varepsilon} dy \le C_\varepsilon |Q|^{-1-\varepsilon}. \tag{8.9}$$

We remark that (8.9) is usually stated in terms of an integral over Q, but in fact the global bound follows from the local one and duality, since by [JK1], [K] the local version of (8.9) and the L^p version of (1.3) yield the estimate

$$|u(A^-_Q)| \le C \sup_{t<0} \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \le C\|g\|_{L^p(\mathbb{R}^n)},$$

where $u(x, t) = \int_{\mathbb{R}^n} k^{x,t}_-(y) g(y) dy$, and p is the dual exponent to $2 + \varepsilon$.

We now note that hypothesis (i) of Theorem 8.5 follows immediately from (8.7) and (8.9). Moreover, (iii) follows immediately from (8.7) and the following well known estimate of Caffarelli, Fabes, Mortola and Salsa [CFMS] (also [K, Lemma 1.3.2, p. 9]):

$$\omega_-^{A_Q^-}(Q) \geq \frac{1}{C}, \tag{8.10}$$

where $\omega_-^{X^-}$ denotes harmonic measure for L at $X^- \in \mathbb{R}^{n+1}_-$.

It remains to verify that b_Q as defined in (8.7) satisfies hypothesis (ii) of Theorem 8.5. To this end, let $(x,t) \in R_Q^+ \equiv Q \times (0, \ell(Q))$. Then, since for fixed $(x,t) \in \mathbb{R}^{n+1}_+$, we have that $\partial_t^2 \Gamma(x,t,\cdot,\cdot)$ is a solution of $Lu = 0$ in $\mathbb{R}^{n+1}_-$,

$$\theta_t b_Q(x) = |Q|\, t \int \partial_t^2 \Gamma(x,t,y,0)\, k_-^{A_Q^-}(y) dy = |Q|\, t\, \partial_t^2 \Gamma(x,t,A_Q^-), \tag{8.11}$$

by Theorem 8.8 (i.e., [JK1]) and uniqueness in (D2) (e.g., Lemma 4.31 (*i*), although of course, uniqueness in the present setting of real symmetric coefficients appears already in [JK1], [K]). Therefore, by (2.3) and translation invariance in t, we have that

$$|\theta_t b_Q(x)| \leq C \frac{t}{\ell(Q)},$$

from which hypothesis (ii) follows readily. Thus, given Theorem 8.5, we conclude that

$$\int_0^\infty \int_{\mathbb{R}^n} |t \partial_t^2 \mathcal{S}_t f(x)|^2 \frac{dxdt}{t} \leq C \|f\|_2^2.$$

The corresponding square function estimate in the lower half-space follows by the same argument, if we replace $k_-^{A_Q^-}$ by $k_+^{A_Q^+}$ in the definition of b_Q. We then obtain (8.2) as desired.

Next, we show that Theorem 8.6 implies (8.1). We consider only the case $t > 0$, the other case being handled by a similar argument. Again, it suffices to construct systems $\{b_Q^1\}$, $\{b_Q^2\}$, now with b_Q^1 and b_Q^2 supported in Q, satisfying hypotheses (i), (ii) and (iii) of Theorem 8.6.

In fact, we shall use the same construction as before, except that we truncate the function outside of Q, i.e. we set

$$b_Q^1 \equiv |Q|\, k_-^{A_Q^-} 1_Q = b_Q 1_Q, \quad b_Q^2 \equiv |Q|\, k_+^{A_Q^+} 1_Q \tag{8.12}$$

As before, (iii) and (i) follow immediately from [CFMS], and (8.9), respectively.

It remains to establish (ii). We observe first that, as in (8.11),

$$|\partial_t \mathcal{S}_t b_Q(x)| = |Q| \left| \int_{\mathbb{R}^n} \partial_t \Gamma(x,t,y,0) k_-^{A_Q^-}(y) dy \right| = |Q|\, |\partial_t \Gamma(x,t,A_Q^-)| \leq C,$$

uniformly in $(x,t) \in \mathbb{R}^{n+1}_+$, where b_Q is defined as in (8.7), and we have used (2.3) and the fact that $t > 0$. We now claim that, for $x \in Q$ and $t > 0$, the same L^∞ bound holds for $\partial_t \mathcal{S}_t(\varphi_Q b_Q)(x)$, where $\varphi_Q \in C_0^\infty$, $\varphi_Q \equiv 1$ on $5Q$, $\operatorname{supp} \varphi_Q \subseteq 6Q$, with $\|\nabla_\| \varphi_Q\|_\infty \leq C/\ell(Q)$. Indeed, fixing $(x,t) \in Q \times (0,\infty)$, and setting $u = \partial_t \Gamma(x,t,\cdot,\cdot)$, we have that

$$\partial_t \mathcal{S}_t(\varphi_Q b_Q)(x) = |Q| \int_{\mathbb{R}^n} u(y,0) \varphi_Q(y) \frac{\partial G^-}{\partial \nu_y^-}(A_Q^-, y, 0) dy.$$

We now extend φ_Q smoothly into the lower half-space so that $\varphi_Q(y,s) \equiv 1$ on $5Q \times (0, -\ell(Q)/4)$, $\varphi_Q(y,s)$ vanishes in $\mathbb{R}^{n+1}_- \setminus [6Q \times (0, -\ell(Q)/2)]$, and

$$\|\nabla \varphi_Q\|_{L^\infty(\overline{\mathbb{R}^{n+1}_-})} \leq C \ell(Q)^{-1}.$$

Since $G^-(A_Q^-, \cdot, \cdot)$ and u are both solutions of $Lu = 0$ in $\operatorname{supp} \varphi_Q$, we obtain from Green's formula (whose use may be justified in the sense of Lemma 4.3 (*iii*)) that

$$\partial_t S_t(\varphi_Q b_Q)(x) = |Q| \iint_{\mathbb{R}^{n+1}_-} A_{y,s} \nabla G^-(A_Q^-, y, s) \nabla \varphi_Q(y, s) u(y, s) dy ds \\ - |Q| \iint_{\mathbb{R}^{n+1}_-} G^- \nabla \varphi_Q \cdot A \nabla u dy ds \equiv I + II.$$

We first consider term II. Let $D_Q \equiv \operatorname{supp} \nabla \varphi_Q$. By the definition of $\varphi_Q(y, s)$, a standard estimate for G^-, and Cauchy-Schwarz, we have that

$$|II| \le C\ell(Q)^{(n+1)/2} \left(\iint_{D_Q} |\nabla \partial_t \Gamma(x, t, y, s)|^2 dy ds \right)^{\frac{1}{2}} \\ \le C\ell(Q)^{(n-1)/2} \left(\iint_{\tilde{D}_Q} |\partial_t \Gamma(x, t, y, s)|^2 dy ds \right)^{\frac{1}{2}},$$

where the last inequality follows by Caccioppoli's inequality, and where $\tilde{D}_Q$ is a fattened version of D_Q. But for $x \in Q$, $t > 0$, and $(y, s) \in \tilde{D}_Q$, we have by (2.3) that

$$|\partial_t \Gamma(x, t, y, s)| \le C|Q|^{-1},$$

hence $|II| \le C$. Similarly,

$$|I| \le C\ell(Q)^{(n-1)/2} \left(\iint_{D_Q} |\nabla G^-(A_Q^-, y, s)|^2 \right)^{\frac{1}{2}} \le C,$$

again by Caccioppoli. Altogether then, $\sup_{t>0} \|\partial_t S_t(\varphi_Q b_Q)\|_{L^\infty(Q)} \le C$, and therefore

$$\sup_{t>0} \int_Q |\partial_t S_t(\varphi_Q b_Q)|^2 \le C|Q|. \tag{8.13}$$

To prove (ii), it will be enough to observe that, for any kernel $K(x, y)$ satisfying (8.3)(a), we have

$$\int_Q \left| \int K(x, y) 1_{6Q \setminus Q}(y) f(y) dy \right|^2 dx \le C \int_{6Q \setminus Q} |f|^2. \tag{8.14}$$

Indeed, given (8.14), we may replace φ_Q by 1_Q in (8.13) (with controlled error), and (ii) follows. The proof of (8.14) is omitted. Since $\Gamma(x, t, y, 0) = \Gamma(y, -t, x, 0)$, a similar argument yields the corresponding bound for $(\partial_t S_t)^{tr}(b_Q^2)$, and (8.1) now follows.

9. P T 1.12:

We now consider invertibility of the layer potentials in the case of real symmetric coefficients. The proof will follow the strategy of Verchota [V], using the well known "Rellich identities" combined with the method of continuity. In our case, the continuity argument will exploit Theorem 1.11.

Proof of Invertibility. From self-adjointness and integration by parts, we obtain the equivalence

$$\|\partial_\nu u\|_{L^2(\mathbb{R}^n)} \approx \|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{9.1}$$

for solutions of $Lu = 0$ in $\mathbb{R}^{n+1}_{\pm}$ for which $\widetilde{N}_*(\nabla u) \in L^2$, where the implicit constants depend only upon ellipticity (see, e.g., [K] for details). In particular, (9.1) holds for $u(\cdot, t) \equiv \mathcal{S}_t f$, with $f \in L^2$. By the jump relation formulae Lemma 4.18, (9.1) becomes

$$\|\left(\pm\frac{1}{2}I + \widetilde{K}\right) f\|_2 \approx \|\nabla_x \mathcal{S}_0 f\|_2. \tag{9.2}$$

Thus, by the triangle inequality and (9.2) we have

$$\|f\|_2 \leq C\|\left(\frac{1}{2}I + \widetilde{K}\right) f\|_2 \tag{9.3}$$

and also

$$\|f\|_2 \leq C\|\nabla_x \mathcal{S}_0 f\|_2, \tag{9.4}$$

where the constants in (9.3) and (9.4) depend only on ellipticity. Moreover, if we set

$$L_\sigma \equiv -\operatorname{div} A_\sigma \nabla, \quad 0 \leq \sigma \leq 1,$$

where

$$A_\sigma \equiv (1-\sigma)\mathbb{I} + \sigma A,$$

and $\mathbb{I}$ denotes the $(n+1)\times(n+1)$ identity matrix, then (9.3) and (9.4) hold, uniformly in σ, for the layer potentials associated to L_σ; indeed, we have uniform control of the ellipticity constants for A_σ. By the result of Section 8, we of course have boundedness of the layer potentials associated to L_σ, again with uniform constants depending only upon ellipticity and dimension. Thus, once we have established invertibility of the layer potentials associated to L_σ, for a given σ, the corresponding Layer Potential Constants will depend only upon ellipticity and dimension, since, in particular, the quantitative bounds for the inverses are precisely the constants in (9.3) and (9.4). We may therefore establish invertibility of $\frac{1}{2}I+\widetilde{K} : L^2 \to L^2$ and $\mathcal{S}_0 : L^2 \to \dot{L}^2_1$ as follows. Since L_0 clearly has Good Layer Potentials, we may invoke Theorem 1.11 to deduce that L_σ has Good Layer potentials, for $0 \leq \sigma < \epsilon_0$, for some ϵ_0 depending only upon ellipticity and dimension. By our previous observation concerning the uniform control of the layer potential constants, we may then iterate this procedure, advancing each time by the same distance ϵ_0, so that we reach $A = A_1$ in finitely many steps. □

10. A :

Suppose that $L = -\operatorname{div} a\nabla$, where a is a constant complex elliptic matrix. Following [FJK], we observe that L has Fourier symbol

$$q(i\xi, i\tau) = \sum_{j,k=1}^{n+1} a_{j,k}\xi_j\xi_k = a_{n+1,n+1}(\tau - \tau_+(\xi))(\tau - \tau_-(\xi)),$$

where $\xi_{n+1} \equiv \tau$, and $\tau_\pm : \mathbb{R}^n \to \mathbb{C}$ are each homogeneous of degree 1, $C^\infty(S^{n-1})$, with

$$\Im m\, \tau_+(\xi) \geq \mu, \quad \Im m\, \tau_-(\xi) \leq -\mu, \tag{10.1}$$

for some $\mu > 0$. In particular,

$$|\tau_+(\xi) - \tau_-(\xi)| \approx |\xi|, \quad \xi \in \mathbb{R}^n. \tag{10.2}$$

The fundamental solution $\Gamma(x,t)$ is a convolution kernel with Fourier symbol $q(i\xi, i\tau)^{-1}$. Inverting the Fourier symbol in t only, and then using the method of residues, we obtain

$$\widehat{\Gamma}(\cdot, t)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it\tau}}{q(i\xi, i\tau)} d\tau = -\frac{e^{it\tau_+(\xi)}1_{\{t>0\}} + e^{it\tau_-(\xi)}1_{\{t<0\}}}{i a_{n+1,n+1}\,(\tau_+(\xi) - \tau_-(\xi))},$$

so by (10.2) and the accretivity of $a_{n+1,n+1}$, we have in particular that

$$|\widehat{\Gamma}(\cdot,0)(\xi)| \approx |\xi|^{-1}.$$

Consequently, $S_0 : L^2 \to \dot{L}^2_1$ is bounded and invertible, by Plancherel's Theorem. One also readily verifies via Plancherel's Theorem that

$$\sup_{t\neq 0} \|\nabla S_t\|_{op} \leq C, \quad |||t\partial_t^2 S_t|||_{op} \leq C.$$

Finally, we note that $f \to \left(\frac{1}{2}I + \widetilde{K}\right) f = \partial_\nu S_t f|_{t=0^+}$ is invertible on L^2. Indeed, the corresponding Fourier symbol is

$$-\lim_{t\to 0^+} e_{n+1} \cdot a\widehat{\nabla\Gamma}(\cdot,t)(\xi) = \frac{a_{n+1,n+1}\tau_+(\xi) + \sum_{j=1}^n a_{n+1,j}\xi_j}{a_{n+1,n+1}\left(\tau_+(\xi) - \tau_-(\xi)\right)},$$

and by [AQ], Lemma 4, the modulus of the numerator $\approx |\xi|$. By the accretivity of $a_{n+1,n+1}$ and (10.2), the same holds for the denominator, and the invertibility follows. Of course, a similar observation holds for $-\frac{1}{2}I + \widetilde{K}$.

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